Toward an Epistemic Foundation for Comparative Confidence

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1 Comparative Confidence and its Formal Representation

The contemporary epistemological literature has been focused, primarily, on two types of judgment: full belief and numerical confidence (*viz.*, credence). Not as much attention has been paid to comparative (or relational) epistemic attitudes.[1] In this part of the book, we will be concerned with relational epistemic attitudes that we will call *comparative confidence* judgments. Specifically, we will analyze two kinds of comparative confidence judgments. We will use the notation $\ulcorner p \succ q \urcorner$ to express a comparative confidence judgment which can be glossed as $\ulcorner S$ is *strictly more confident* in the truth of p than they are in the truth of $q\urcorner$, and we will use the notation $\ulcorner p \sim q \urcorner$ to express a comparative confidence judgment which can be glossed as $\ulcorner S$ is *equally confident* in the truth of p and the truth of $q\urcorner$. For the purposes of this investigation, we will make some simplifying assumptions about $\succ$ and $\sim$. These simplifying assumptions are not essential features of our framework, but they will make it easier to develop and explain our justifications of various epistemic coherence requirements for comparative confidence relations.[2]

Our first simplifying assumption is that our agents S form judgments regarding (pairs of) propositions drawn from a finite Boolean algebra of (n) propositions $\mathcal{B}_n$. More precisely, our agents will make comparative confidence judgments regarding

[1] It wasn't always thus. **?**, **??**, **?**, **?** and **?** emphasized the importance (and perhaps even fundamentality) of comparative confidence. Moreover, there are some notable recent exceptions to this general trend (**?**).

[2] It is difficult to articulate the intended meanings of $\ulcorner p \succ q \urcorner$ and $\ulcorner p \sim q \urcorner$ without implicating that these relations reduce to (or essentially involve) some comparison of *non-relational* (and perhaps numerical) credences $b(\cdot)$ of the agent S [*e.g.*, $b(p) > b(q)$ or $b(p) = b(q)$]. But, it is important that no such reductionist assumption be made in the present context. Later in this chapter, we will discuss issues of numerical representability of $\succeq$-relations. And, we will discuss some problems involving numerical reductionism for comparative confidence in the Negative Phase. But, the reader should assume that $\succ$ and $\sim$ are *autonomous* relational attitudes, which may not (ultimately) reduce to (or essentially involve) any non-relational attitude of agents. Other glosses on $\ulcorner p \succ q \urcorner$ ($\ulcorner p \sim q \urcorner$) have been given in the literature, *e.g.*, $\ulcorner S$ judges p to be strictly more believable/plausible than $q\urcorner$ ($\ulcorner S$ judges p and q to be equally believable/plausible$\urcorner$). Another gloss of $\ulcorner p \succ q \urcorner$ ($\ulcorner p \sim q \urcorner$) is $\ulcorner S$'s total evidence strictly favors p over $q\urcorner$ ($\ulcorner S$'s total evidence favors neither p over q nor q over $p\urcorner$).

(pairs of) propositions on m-proposition *agendas* $\mathcal{A}$, which are (possibly proper) subsets of $\mathcal{B}_n$ (*viz.*, $m \leq n$). Because we are assuming that the objects of comparative confidence judgments are (pairs of) classical, possible-worlds propositions, we will (as in Part I) be presupposing a kind of (weak) logical omniscience, according to which the the agent is aware of all logical equivalences (and so we may always substitute logical equivalents within comparative confidence judgments).

Next, we adopt some conventions regarding the order-structure of an agent's comparative confidence relation. First, we will assume that the relation $\succ$ constitutes a *strict order* on the agenda in question $\mathcal{A}$. That is, we will assume that $\succ$ satisfies the following two ordering conditions.

Irreflexivity of $\succ$. For all $p \in \mathcal{A}$, $p \not\succ p$.

Transitivity of $\succ$. For all $p, q, r \in \mathcal{A}$, *if* $p \succ q$ and $q \succ r$, *then* $p \succ r$.

Second, we assume that $\sim$ is an *equivalence relation* on $\mathcal{A}$. That is, we assume:

Reflexivity of $\sim$. For all $p \in \mathcal{A}$, $p \sim p$.

Transitivity of $\sim$. For all $p, q, r \in \mathcal{A}$, *if* $p \sim q$ and $q \sim r$, *then* $p \sim r$.

Symmetry of $\sim$. For all $p, q \in \mathcal{A}$, *if* $p \sim q$, *then* $q \sim p$.

Just as in the case of full belief, we will be focusing on agendas $\mathcal{A}$ over which our agents are *opinionated*. More precisely, we will be making the following opinionation assumption regarding our agents' comparative confidence relations.[3]

Opinionation of $\succeq$. For each of the $\binom{m}{2}$ pairs of propositions $p, q \in \mathcal{A}$, our agents will form *exactly one* of the following three possible comparative confidence judgments: *either* $p \succ q$ *or* $q \succ p$ *or* $p \sim q$.

We will use the symbol $\succeq$ to refer to an agent's entire comparative confidence relation over $\mathcal{A}$, *i.e.*, the set of her comparative confidence judgments over $\mathcal{A}$. If $\succeq$ satisfies all five of the ordering assumptions regarding $\succ$ and $\sim$ above, then we will say that $\succeq$ is a *total preorder* on $\mathcal{A}$. We will adopt the notation $\ulcorner p \succeq q \urcorner$ to refer to a (generic) individual comparative confidence judgment regarding p and q, which may be either a strict ($\succ$) or an indifference ($\sim$) judgment.

[3] We are well aware of the fact that some of these assumptions about the order structure of $\succeq$ (especially, transitivity and opinionation) have been a source of controversy in the literature on coherence requirements for comparative confidence relations. See, for instance, (**???**) for discussion. However, our local, agenda-relative versions of these assumptions are not as controversial as the usual, global versions, which apply to the entire algebra $\mathcal{B}_n$. Having said that, even our agenda-relative ordering assumptions remain somewhat controversial. But, we have chosen (in this Positive Phase) to simplify things by bracketing controversies about the order structure of $\succeq$ on finite agendas $\mathcal{A}$. We will return to some of these questions regarding the order-structure of comparative confidence relations $\succeq$ in the Negative Phase.

Two final notes are in order here, by way of setup. The combination of (weak) logical equivalence and our ordering assumptions entails the following non-trivial logical omniscience principle, which will play a key role in our proofs below.

(LO) If p and q are logically equivalent, then S judges that $p \sim q$. [And, if S judges $p \succ q$, then p and q are *not* logically equivalent.]

Furthermore, we will add the following final background assumption about the kinds of orderings we will be evaluating.

Regularity. For all $p \in \mathcal{A}$, if p is contingent, then $p \succ \bot$ and $\top \succ p$.[4]

Our aim here will be to provide direct, epistemic justifications for various formal coherence requirements — above and beyond the assumption that $\succeq$ is a (Regular) total preorder — that have been proposed for $\succeq$ in the contemporary literature. But, first, we'll need to explain how we're going to formally represent $\succeq$-relations.

One convenient way to represent a $\succeq$-relation on a finite agenda $\mathcal{A}$ containing m propositions is *via* its *adjacency matrix*. Let $p_1, \ldots, p_m$ be the m propositions contained in some agenda $\mathcal{A}$. The adjacency matrix $A^{\succeq}$ of a $\succeq$-relation on $\mathcal{A}$ is an $m \times m$ matrix of 0s and 1s such that $A^{\succeq}_{ij} = 1$ iff $p_i \succeq p_j$.

It's instructive to look at a simple example. Consider the simplest sentential Boolean algebra $\mathcal{B}_4$, which is generated by a single contingent claim P. This algebra $\mathcal{B}_4$ contains the following four propositions: $\langle p_1, p_2, p_3, p_4 \rangle = \langle \top, P, \neg P, \bot \rangle$. To make things concrete, let P be the claim that a coin (which is about to be tossed) will land heads (so, $\neg P$ says that the coin will land tails). Suppose our agent S is equally confident in (*viz.*, epistemically indifferent between) P and $\neg P$. And, suppose that S is strictly more confident in $\top$ than in any of the other propositions in $\mathcal{B}_4$, and that S is strictly less confident in $\bot$ than in any of the other propositions in $\mathcal{B}_4$. This description fully characterizes a (regular, totally preordered) $\succeq$-relation on the agenda consisting of the entire Boolean algebra $\mathcal{B}_4$, which has the adjacency matrix representation (and the graphical representation) depicted in Figure 1. In the adjacency matrix $A^{\succeq}$ of $\succeq$, a 1 appears in the $\langle i, j \rangle$-th cell just in case $p_i \succeq p_j$. In the graphical representation of $\succeq$, an arrow is drawn from p_i to p_j only if[5] $p_i \succeq p_j$. With our basic formal framework in hand, we are ready to proceed.

In the next section, we'll discuss a fundamental coherence requirement for $\succeq$ that has been (nearly) universally accepted in the contemporary literature. Then, we will layout our general framework for grounding $\succeq$-coherence requirements, and we will

[4] In the case of numerical credence, the assumption of (numerical) Regularity (**?**) has been a source of significant controversy (**?**). However, our present assumption of *comparative* Regularity is significantly less controversial than its numerical analogue (**?**). So, our present assumption of Regularity is not something that most people would find objectionable. We will discuss the relationship between qualitative and quantitative Regularity assumptions in the Negative Phase below. Finally, it is worth noting that in the case of numerical confidence (Part III) no assumption of (numerical) Regularity will be required to apply our framework in a probative way.

[5] We're omitting some arrows in our graphical representations of total preorders. Specifically, because such $\succeq$-relations are transitive, there will be lots more arrows than we're showing in our diagrams. So, we're just using these graphical representations to give the gist of the relation depicted.

$\succeq$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	1	1	1
$\bot$	0	0	0	1

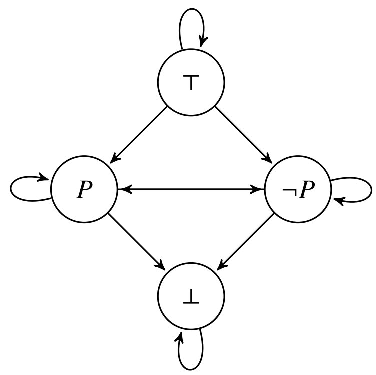

Figure 1: Adjacency matrix $A^{\succeq}$ and graphical representation of an intuitive $\succeq$-relation on the smallest Boolean algebra $\mathcal{B}_4$

explain how our framework can be used to provide a novel epistemic justification of this fundamental coherence requirement for $\succeq$.

2 The Fundamental Coherence Requirement for $\succeq$

The literature on coherence requirements for $\succeq$ has become rather extensive. A plethora of coherence requirements of varying degrees of strength, *etc.*, have been proposed and defended. We will not attempt to survey all of these requirements here.[6] Instead, we will focus on a particular family of requirements, which can be expressed *via* both axiomatic constraints on $\succeq$ and in terms of various kinds of numerical representability of $\succeq$. We begin with the most fundamental of the existing coherence requirements, which is common to almost all the accounts of comparative confidence we have seen. In order to properly introduce this fundamental coherence requirement for comparative confidence, we will first need to introduce the concept of a *plausibility measure* (or a *capacity*) on a Boolean algebra.

A *plausibility measure* (*a.k.a.*, a *capacity*) on a Boolean algebra $\mathcal{B}_n$ is real-valued function $\text{Pl} : \mathcal{B}_n \mapsto [0, 1]$ which maps propositions from $\mathcal{B}_n$ to the unit interval, and which satisfies the following three axioms (**?**, *p.* 51).

(Pl_1) $\text{Pl}(\bot) = 0$.

(Pl_2) $\text{Pl}(\top) = 1$.

(Pl_3) For all $p, q \in \mathcal{B}_n$, *if* p entails q, *then* $\text{Pl}(q) \geq \text{Pl}(p)$.

That is, a plausibility measure $\text{Pl}(\cdot)$ is a real-valued function from $\mathcal{B}_n$ to the unit interval, which (a) assigns maximal value to tautologies, (b) assigns minimal value to

[6]See (**?**, Ch. 2) for an up-to-date and comprehensive survey. See, also, (**??**) and references therein.

contradictions, and (c) never assigns logically stronger propositions a greater value than logically weaker propositions. The fundamental coherence requirement for $\succeq$ — which we will call ($\mathfrak{C}$) — can be stated in terms of representability by a plausibility measure. That is, here is one way of stating ($\mathfrak{C}$).

($\mathfrak{C}$) It is a requirement of ideal epistemic rationality that an agent's $\succeq$-relation (assumed to be a regular, total preorder on a finite agenda $\mathcal{A}$) be *representable by some plausibility measure*. That is, a $\succeq$-relation is *coherent only if* there is some plausibility measure Pl such that for all $p, q \in \mathcal{A}$

$$p \succ q \text{ iff } \mathrm{Pl}(p) > \mathrm{Pl}(q), \text{ and } p \sim q \text{ iff } \mathrm{Pl}(p) = \mathrm{Pl}(q).$$ [7]

It is well known (**?**) that ($\mathfrak{C}$) can be also stated *via* the following axiomatic constraints on the relation $\succeq$.

($\mathfrak{C}$) It is a requirement of ideal epistemic rationality that an agent's $\succeq$-relation (assumed to be a regular, total preorder on a finite agenda $\mathcal{A}$) satisfy the following two axiomatic constraints

(A_1) $\top \succ \bot$.

(A_2) For all $p, q \in \mathcal{A}$, *if* p entails q, *then* $p \not\succ q$.

In other words, ($\mathfrak{C}$) requires (A_1) that an agent's $\succeq$-relation ranks tautologies *strictly above* contradictions, and (A_2) that an agent's $\succeq$-relation lines up with the "is deductively entailed by" (*viz.*, the "logically follows from") relation.

As far as we know, despite the (nearly) universal acceptance of ($\mathfrak{C}$) as a coherence requirement for $\succeq$, no *epistemic* justification has been given for ($\mathfrak{C}$). Various *pragmatic* justifications of requirements like ($\mathfrak{C}$) have appeared in the literature. For instance, various "Money Pump" arguments and "Representation Theorem" arguments (**????**) have aimed to show that agents with $\succeq$-relations which *violate* ($\mathfrak{C}$) must exhibit some sort of "pragmatic defect". Once again, following Joyce (**??**), we will be focusing on *non*-pragmatic (*viz.*, *epistemic*) defects implied by the synchronic incoherence of an agent's $\succeq$-relation. Specifically, we will (as in the case of full belief above) be focusing on *alethic* and *evidential* evaluations of an agent's $\succeq$-relation (over a given, finite agenda $\mathcal{A}$), which we take to be *distinctively epistemic*.

In the next section, we'll explain our general (broadly Joycean) strategy for grounding epistemic coherence requirements for $\succeq$. This will allow us to explain *why* ($\mathfrak{C}$) is a requirement of ideal *epistemic* rationality. Moreover, our explanation will be a unified

[7] Strictly speaking, these traditional coherence requirements should be stated *via* axioms (or f-representability constraints) that a relation satisfies *over an entire algebra* $\mathcal{B}_n$. So, strictly speaking, we should be talking about whether a relation $\succeq$ on an agenda $\mathcal{A}$ is *extendible to a relation on the entire algebra* $\mathcal{B}_n$ which satisfies the axioms/representability requirements in question. And, it is extendibility claims such as these that our proofs will establish (or refute). However, for ease of exposition, we will state requirements in terms of constraints on $\succeq$ *over the agenda* $\mathcal{A}$ in question.

and principled one, which dovetails nicely with similar explications of formal, epistemic coherence requirements for other types of judgment (*e.g.*, as explained in Parts I and III of this book).

3 Grounding the Fundamental Coherence Requirement for $\succeq$

In this section, we show how a natural generalization of Joyce's argument for probabilism can be used to provide a compelling epistemic justification of the fundamental coherence requirement for $\succeq$ [($\mathfrak{C}$)]. This involves going through the Joycean "three steps" (as in §**??** of Part I above) — as applied to $\succeq$.

3.1 Step 1: Qualitative inaccuracy of $\succeq$-judgments

We will adopt the (broadly Joycean) idea that a confidence *ordering* is (qualitatively) inaccurate (at w) *iff*[8] it fails to rank all the truths *strictly above* all the falsehoods (at w). Two facts about the inaccuracy (or alethic defectiveness) of *individual* comparative confidence judgments follow immediately from this fundamental assumption about the (qualitative) inaccuracy of comparative confidence orderings.

> **Fact 1**. If $q \;\&\; \neg p$ is true at w, then $p \succ q$ is inaccurate at w.
>
> **Fact 2**. If $p \not\equiv q$ is true at w, then $p \sim q$ is inaccurate at w.

The unifying idea behind these two facts is that an individual comparative confidence judgment $p \succeq q$ is *inaccurate/incorrect* at w iff $p \succeq q$ *implies that the ordering* $\succeq$ *of which it is a part is qualitatively inaccurate at w (i.e., that some truth fails to be ranked strictly above some falsehood by the relation* $\succeq$ *at w).* The two facts above identify the two cases in which this alethic defect is manifest.

Assuming *extensionality*, *i.e.*, that the (qualitative and quantitative) inaccuracy of a comparative confidence ordering over an agenda $\mathcal{A}$ is determined solely by the truth values of the propositions in $\mathcal{A}$ at w, our approach to Step 1 is quite natural (but see *fn.* 8). And, given our particular extensional approach to qualitative $\succeq$-inaccuracy, the two facts above are the *only* (qualitative) facts regarding the (qualitative) inaccuracy of *individual* comparative confidence judgments $p \succeq q$.

Nonetheless, the type of mistake identified in Fact 1 seems *worse* than the type of mistake identified in Fact 2. For the mistake identified in Fact 1 implies that some falsehoods are ranked *strictly above* some truths by the ordering $\succeq$ (in w), whereas the

[8] Joyce would probably not want to accept the only if direction of this biconditional. That is, Joyce would likely want to adopt a weaker notion of qualitative $\succeq$-*in*accuracy, according to which a confidence ordering is inaccurate iff it fails to be identical to the (unique) ordering $\mathring{\succeq}_w$ at w which not only ranks all truths above all falsehoods, but *also ranks propositions with the same truth-value at the same level*. Unfortunately, there is no evidentially proper inaccuracy measure that suits this weaker ($\mathring{\succeq}_w$ non-identity) notion of qualitative $\succeq$-inaccuracy. It is for this reason that we adopt our stronger sense of "inaccurate". We will return to this negative result in the Negative Phase.

mistake identified in Fact 2 implies only that the truths and falsehoods are *not fully separated* by the ordering $\succeq$ (in w). We will return to this difference between these two kinds of comparative inaccuracies in the next section, when we discuss *quantitative measures of inaccuracy* for $\succeq$-judgments (and sets thereof).

One more remark is in order about the distinction between these two types of comparative inaccuracies. There is something (*prima facie*) peculiar about the type of mistake identified in Fact 2. Recall the relation $\succeq$ depicted in Figure 1, above. In that case, we have an agent who is indifferent between the claim that the coin will land heads and the claim that the coin will land tails ($P \sim \neg P$). According to Fact 2, this is *automatically* an *inaccurate* comparative judgment, since $P \equiv \neg P$ is a *logical falsehood*. This may seem somewhat odd, since this relation could certainly be *supported by the agent's evidence* (*e.g.*, if their evidence consisted solely of the claim that the coin is *fair*). What this reveals is that, unlike the case of full belief, in the case of comparative confidence, there can be individual judgments which are supported by some body of evidence — despite the fact that they are (qualitatively) inaccurate in every possible world.[9] We'll return to the distinction between alethic and evidential norms for comparative confidence judgments below. Meanwhile, with our explication of the (qualitative) inaccuracy of individual comparative confidence judgments in hand, we're ready to proceed to Step 2.

3.2 Step 2: Measuring Quantitative $\succeq$-Inaccuracy

Our next step involves the explication of a (point-wise, additive) measure of (total) inaccuracy for comparative confidence relations (or judgment sets). By analogy with the case of full belief, we could opt for a naïve, *mistake-counting* measure of the overall inaccuracy of a comparative confidence relation (*viz.*, judgment set) $\succeq$. That is, we could opt for

$$\Delta(\succeq, w) \;\; \text{the number of inaccurate individual judgments in} \succeq \text{(at } w\text{)}.$$

To understand Δ, it is helpful to recall that a relation $\succeq$ is just a *set* of $\binom{m}{2}$ pairwise comparative confidence judgments, *i.e.*, a set containing *exactly one* of $\{p \succ q,\, q \succ p,\, p \sim q\}$ for each of the $\binom{m}{2}$ pairs of propositions p and q in the m-proposition agenda $\mathcal{A}$. In each such set, there will be some number (possibly zero) of inaccurate individual comparative confidence judgments (at w). That number is $\Delta(\succeq, w)$.

We do not think $\Delta(\succeq, w)$ is an appropriate inaccuracy measure (for present purposes). We have two main reservations regarding the use of $\Delta(\succeq, w)$.

- As we explained above, the type of mistake identified in Fact 1 seems *worse* (from an alethic point of view) than the type of mistake identified in Fact 2. But, $\Delta(\succeq, w)$ scores both types of mistakes *equally*.[10]

[9] As we'll see in Part III, the same will be true (but *even more starkly*) for numerical credences.

[10] This is disanalogous to the case of full belief, where the two types of mistakes (believing falsehoods and disbelieving truths) seem to be *on a par*. And, this explains why we didn't complain about the fact that our point-wise score i scored *those* two types of mistakes equally (and it also explains, in part, why our naïve, mistake-counting measure $\mathcal{I}(\mathbf{B}, w)$ was evidentially proper).

- Because $\Delta(\succeq, w)$ does not weight the type of mistake identified in Fact 1 *more heavily* than the type of mistake identified in Fact 2, $\Delta(\succeq, w)$ fails to be *evidentially proper* (in the sense of Definition **??**). Consequently, adopting $\Delta(\succeq, w)$ as our measure of inaccuracy would *run afoul* of the evidential requirements we'll be adopting below.

It is helpful here to work through a couple of simple examples involving $\Delta(\succeq, w)$ in order to (begin to) appreciate these two problems. Recall our toy agent who forms judgments on an agenda $\mathcal{A}$ consisting of the (entire) simplest Boolean algebra $\mathcal{B}_4$. Let S's $\succeq$-relation be given by the ordering depicted in Figure 1. There are only two salient possible worlds in this case (w_1 in which P is false, and w_2 in which P is true). It is straightforward to calculate $\Delta(\succeq, w_i)$ in each of these two salient possible worlds. In both worlds, the relation $\succeq$ contains *exactly one* inaccurate indifference judgment: $P \sim \neg P$. And, in both worlds, the relation $\succeq$ does not rank any falsehoods *above* any truths. So, *none* of $\succeq$'s *strict* comparisons is inaccurate in either world. This means that, in both worlds w_i, $\Delta(\succeq, w_i) = 1$. Next, let's consider an *alternative* comparative confidence relation ($\succeq'$) on $\mathcal{B}_4$, as depicted in Figure 2. The only difference between $\succeq$

$\succeq'$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	0	1	1
$\bot$	0	0	0	1

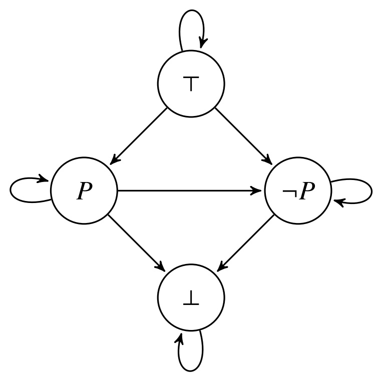

Figure 2: Adjacency matrix $A^{\succeq'}$ and graphical representation of an alternative relation ($\succeq'$) on the smallest Boolean algebra $\mathcal{B}_4$

and $\succeq'$ is that $\succeq'$ ranks P *strictly above* $\neg P$, whereas $\succeq$ is *indifferent* between P and $\neg P$. It is straightforward to compare the values of $\Delta(\succeq, w_i)$ and $\Delta(\succeq', w_i)$ in the two salient possible worlds.

$$\Delta(\succeq, w_1) = 1 = \Delta(\succeq', w_2)$$

$$\Delta(\succeq, w_2) = 1 > 0 = \Delta(\succeq', w_2)$$

As you can see, $\succeq'$ is *never more* inaccurate than $\succeq$, according to Δ; and, $\succeq'$ is *sometimes less inaccurate* than $\succeq$, according to Δ (*i.e.*, in world w_2). In other words, according to Δ, $\succeq'$ *weakly dominates* $\succeq$ in overall inaccuracy. This will mean that (given the initial choice of fundamental epistemic principle that we're going to make in Step 3 below) using Δ would have the effect of *ruling out* $\succeq$ as *epistemically irrational*. And, this is *not a good thing*, since $\succeq$ could (intuitively) be supported by S's evidence (*e.g.*, if her evidence consists of the claim that the coin is *fair*). We will delve further into these (probabilistic) evidential requirements for $\succeq$, below.

Presently, our task is to introduce an alternative measure of $\succeq$-inaccuracy, which avoids both of these problems encountered by Δ. As we will explain later (in the section on evidential requirements for comparative confidence relations), there is (essentially) *only one way* to fix the two problems that plague Δ. Assuming (as a matter of convention) that we assign an inaccuracy score of 1 to the types of ($\sim$) mistakes identified in Fact 2, there is *only one way* to score the types of ($\succ$) mistakes of identified in Fact 1 that leads to an evidentially proper inaccuracy measure, and that is to assign them a score that is *exactly twice as large*. In other words, the following point-wise measure is (essentially) *uniquely* determined by the constraint that the resulting measure of total inaccuracy be evidentially proper.[11]

$$\mathfrak{i}_{\succeq}(p \succeq q, w) \begin{cases} 2 & \text{if } q \;\&\; \neg p \text{ is true in } w, \text{ and } p \succ q, \\ 1 & \text{if } p \not\equiv q \text{ is true in } w, \text{ and } p \sim q, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{C} = \{c_1, \ldots, c_{\binom{m}{2}}\}$ be the comparative confidence judgment set determined by a relation $\succeq$ on an m-proposition agenda $\mathcal{A}$. That is, for each pair $p, q \in \mathcal{A}$, $\mathbf{C}$ contains exactly one of the three comparative judgments $p \succ q$, $q \succ p$ or $p \sim q$. In the section on evidential requirements for comparative confidence relations, we will prove that (a) $\mathfrak{i}_{\succeq}(p \succeq q, w)$ undergirds an evidentially proper measure[12]

$$\mathcal{I}_{\succeq}(\succeq, w) \;\; \sum_i \mathfrak{i}_{\succeq}(c_i, w)$$

of the (total) inaccuracy of a relation $\succeq$ (on $\mathcal{A}$), and (b) essentially no other scoring scheme for the two types of mistaken comparative confidence judgments identified in Facts 1 and 2 yields an evidentially proper (total) inaccuracy measure for $\succeq$. For now, let's just run with our point-wise and total measures $\mathfrak{i}_{\succeq}$ and $\mathcal{I}_{\succeq}$ above.

[11] If we were to adopt the stronger notion of "accurate confidence ordering" mentioned in *fn.* 8, then there would be *no* evidentially proper measure of inaccuracy (see the Negative Phase). It is important to note that this result trades essentially on the assumption that our overall inaccuracy measures are *additive*. **?** is investigating the use of non-additive measures of inaccuracy for comparative confidence judgment sets, in an attempt to apply the framework to the weaker Joycean notion of "inaccuracy" (or non-vindication) of comparative confidence relations.

[12] We will use the notation $\mathcal{I}_{\succeq}(\succeq, w)$ to refer to the total inaccuracy of a relation $\succeq$ on an agenda $\mathcal{A}$. We may also use the notation $\mathcal{I}_{\succeq}(\mathbf{C}, w)$ to refer to the same measure, where it is understood that $\mathbf{C}$ is the set of individual comparative confidence judgments (on $\mathcal{A}$) that comprise the relation $\succeq$.

3.3 Step 3: The fundamental epistemic principle

We will begin with the same fundamental epistemic principle with which we began in Part I — weak accuracy dominance avoidance.[13]

> **Weak Accuracy-Dominance Avoidance for Comparative Confidence Relations** (WADA$_{\succeq}$). $\succeq$ should *not be weakly dominated* in accuracy. Or, to put this more formally (in terms of $\mathcal{I}_{\succeq}$), there should *not* exist a binary relation[14] $\succeq'$ on $\mathcal{A}$ such that both
>
> (i) $(\forall w)\,[\mathcal{I}_{\succeq}(\succeq', w) \leq \mathcal{I}_{\succeq}(\succeq, q)]$, and
>
> (ii) $(\exists w)\,[\mathcal{I}_{\succeq}(\succeq', w) < \mathcal{I}_{\succeq}(\succeq, w)]$.

Let's return to our toy example(s) above, to illustrate $\mathcal{I}_{\succeq}$-*non*-dominance. Recall our toy agent S who has formed comparative confidence judgments ($\succeq$) on $\mathcal{B}_4$, regarding the toss of a coin (as depicted in Figure 1). It can be shown (more on this later) that there is *no* binary relation $\succeq'$ on $\mathcal{B}_4$ that weakly dominates S's $\succeq$ in accuracy (according to $\mathcal{I}_{\succeq}$). We will prove this general claim later on. Meanwhile, we will show that the alternative relation $\succeq'$ (which weakly Δ-dominated $\succeq$) does not weakly $\mathcal{I}_{\succeq}$-dominate $\succeq$. It is easy to see that

$$\mathcal{I}_{\succeq}(\succeq, w_1) = 1 < 2 = \mathcal{I}_{\succeq}(\succeq', w_1)$$

$$\mathcal{I}_{\succeq}(\succeq, w_1) = 1 > 0 = \mathcal{I}_{\succeq}(\succeq', w_2)$$

That is, according to $\mathcal{I}_{\succeq}$, $\succeq$ is more accurate than $\succeq'$ in w_1 but less accurate than $\succeq'$ in w_2. So, neither of these comparative confidence relations weakly dominates the other in accuracy, according to $\mathcal{I}_{\succeq}$.[15]

The avoidance of weak dominance in doxastic inaccuracy is a basic principle of epistemic utility theory. But, as in the case of full belief (see §**??** of Part I), some of our results continue to hold, even if we adopt *weaker and more sacrosanct* fundamental epistemic principles than (WADA$_{\succeq}$). As in Part I, we will discuss two requirements that are weaker than (WADA$_{\succeq}$). Predictably, the first is

[13] In the credal case, weak and strict accuracy dominance avoidance turn out to be (practically) equivalent (see *fn.* **??**). As we saw in Part I, in the case of full belief, (WADA) is strictly stronger than (SADA) and (SSADA). Similarly, when it comes to comparative confidence, (WADA$_{\succeq}$) will be strictly stronger than (SADA$_{\succeq}$) and (SSADA$_{\succeq}$), both of which will also be discussed below.

[14] Note that we do not restrict this quantifier to total preorders. If there is *any* binary relation $\succeq'$ on that weakly accuracy dominates $\succeq$ on $\mathcal{A}$, then we will take this to be an *alethic defect* of $\succeq$.

[15] In fact, it can be shown that neither of the relations ($\succeq$ or $\succeq'$) discussed in this section is weakly $\mathcal{I}_{\succeq}$-dominated by *any* binary relation on $\mathcal{B}_4$. This will follow from our proof that $\mathcal{I}_{\succeq}$ is evidentially proper, below. Intuitively, this is as it should be. After all, either of these relations *could be* supported by a (rational) agent's total evidence E. For instance, $\succeq$ would be supported by E if E entailed (only) the claim that the coin is *fair*, while $\succeq'$ would be supported by E if E entailed (only) the claim that the coin is *biased toward heads* (**?**, *pp.* 282–3). We will say more about the nature of evidentially supported comparative confidence relations in §5, below.

Strict Accuracy-Dominance Avoidance for Comparative Confidence Relations (SADA$_{\succeq}$). $\succeq$ should *not be strictly dominated* in accuracy. Or, to put this more formally (in terms of $\mathcal{I}_{\succeq}$), there should *not* exist a binary relation $\succeq'$ on $\mathcal{A}$ such that

$$(\forall w)\,[\mathcal{I}_{\succeq}(\succeq', w) < \mathcal{I}_{\succeq}(\succeq, w)].$$

And, the second is the $\succeq$-analogue of (SSADA). Let $\mathbf{M}(\succeq, w)$ denote the set of (qualitatively) *inaccurate* comparative judgments (of either of the two types discussed above) made by a comparative confidence relation $\succeq$ at a possible world w. And, consider the following *bedrock* fundamental epistemic principle.

Strong Strict Accuracy-Dominance Avoidance for Comparative Confidence Relations (SSADA$_{\succeq}$). $\succeq$ should *not be strongly strictly dominated* in accuracy. Formally (in terms of $\mathbf{M}$), there should *not* exist a binary relation $\succeq'$ on $\mathcal{A}$ such that

$$(\forall w)\,[\mathbf{M}(\succeq', w) \subset \mathbf{M}(\succeq, w)].$$

(SADA$_{\succeq}$) entails (SSADA$_{\succeq}$), but not conversely. This is because violating (SSADA$_{\succeq}$) entails that there exists a binary relation $\succeq'$ which *not only strictly dominates* $\succeq$, but *also never makes any mistakes that* $\succeq$ *doesn't already make*. Clearly, if S's $\succeq$ violates (SSADA$_{\succeq}$), this means S is failing to live up to her epistemic aim of making accurate judgments (no matter how this is construed). Other choices of fundamental epistemic principle could be made. But, in this initial investigation, we will stick with (WADA$_{\succeq}$), (SADA$_{\succeq}$) and (SSADA$_{\succeq}$) as our three fundamental (alethic) epistemic principles.[16] In fact, mainly, we'll be making use of (WADA$_{\succeq}$). But, occasionally, we'll point out when certain claims follow from the weaker principles (SADA$_{\succeq}$) or (SSADA$_{\succeq}$). The following theorem undergirds an epistemic justification of ($\mathfrak{C}$).

Theorem 1. *If* $\succeq$ *violates* ($\mathfrak{C}$)*, then* $\succeq$ *violates* (WADA$_{\succeq}$)*, i.e.,* (WADA$_{\succeq}$) $\Rightarrow$ ($\mathfrak{C}$).

In the next section, we look beyond ($\mathfrak{C}$) to stronger coherence requirements for $\succeq$ that have appeared in the literature. As we'll see, (WADA$_{\succeq}$) can be used to provide epistemic justifications for an important family of traditional coherence requirements for $\succeq$.

4 Beyond ($\mathfrak{C}$) — A Family of Traditional Coherence Requirements for $\succeq$

The fundamental requirement ($\mathfrak{C}$) is but one among many coherence requirements that have been proposed for $\succeq$. We will not attempt to survey all of the requirements that

[16] In §5 below, we will discuss *evidential* epistemic requirements and explain how they relate to our alethic requirements (WADA$_{\succeq}$), (SADA$_{\succeq}$) and (SSADA$_{\succeq}$).

have been discussed in the literature (**?**). We'll focus on one particular family of requirements. Before stating the other requirements in the family, we first need to define two more types of numerical functions that will serve as *representers* of comparative confidence relations.

A *mass function* on a Boolean algebra $\mathcal{B}_n$ is real-valued function $m : \mathcal{B}_n \mapsto [0,1]$ which maps propositions from $\mathcal{B}_n$ to the unit interval, and which satisfies the following two axioms.

(M_1) $m(\bot) = 0$.

(M_2) $\sum_{p \in \mathcal{B}_n} m(p) = 1$.

A *belief function* — sometimes called a *Dempster-Shafer function* (**???**) — on a Boolean algebra $\mathcal{B}_n$ is a real-valued function $\text{Bel} : \mathcal{B}_n \mapsto [0,1]$ which maps propositions from $\mathcal{B}_n$ to the unit interval, and which is generated by an underlying mass function m in the following way

$$\text{Bel}_m(p) \quad \sum_{\substack{q \in \mathcal{B}_n \\ q \text{ entails } p}} m(q).$$

It is easy to show that all belief functions are plausibility functions (but not conversely). In this sense, the concept of a belief function is a refinement of the concept of a plausibility function. The class of Belief functions, in turn, contains the class of *probability* functions, which can be defined in terms of a special type of mass function. Let $\mathfrak{s} \in \mathcal{B}_n$ be the *states* of a Boolean algebra $\mathcal{B}_n$ (or, if you prefer, the *state descriptions* of a propositional language $\mathcal{L}$ which generates $\mathcal{B}_n$). The states of $\mathcal{B}_n$ just correspond to the *possible worlds* that are involved in our epistemic evaluations of the agent in question. A *probability* mass function is real-valued function $\mathfrak{m} : \mathcal{B}_n \mapsto [0,1]$ which maps *states* of $\mathcal{B}_n$ to the unit interval, and which satisfies the following two axioms.

($\mathfrak{M}_1$) $\mathfrak{m}(\bot) = 0$.

($\mathfrak{M}_2$) $\sum_{\mathfrak{s} \in \mathcal{B}_n} \mathfrak{m}(\mathfrak{s}) = 1$.

A *probability function* on a Boolean algebra $\mathcal{B}_n$ is a real-valued function $\Pr : \mathcal{B}_n \mapsto [0,1]$ which maps propositions from $\mathcal{B}_n$ to the unit interval, and which is generated by an underlying probability mass function $\mathfrak{m}$ in the following way

$$\Pr_{\mathfrak{m}}(p) \quad \sum_{\substack{\mathfrak{s} \in \mathcal{B}_n \\ \mathfrak{s} \text{ entails } p}} \mathfrak{m}(\mathfrak{s}).$$

It is easy to show that all probability functions are belief functions (but not conversely). So, probability functions are special kinds of belief functions (and belief functions are, in turn, special kinds of plausibility measures).

There are various senses in which a real-valued function f may be said to *represent* a comparative confidence relation $\succeq$. We have already seen the strongest variety of numerical representation, which is called *full agreement*.

Definition 1. *f **fully agrees** with a comparative confidence relation $\succeq$ just in case, for all $p, q \in \mathcal{A}$, $p \succ q$ iff $f(p) > f(q)$, and $p \sim q$ iff $f(p) = f(q)$.*

Thus, the fundamental coherence requirement ($\mathfrak{C}$) requires that there exist a plausibility measure Pl which *fully agrees* with $\succeq$. Another, weaker kind of numerical representability is called *partial agreement*.

Definition 2. *f **partially agrees** with a comparative confidence relation $\succeq$ just in case, for all $p, q \in \mathcal{A}$, $p \succ q$ only if $f(p) > f(q)$.*

If f partially agrees with $\succeq$, then we will say that f *partially represents* $\succeq$. And, if f fully agrees with $\succeq$, then we will say that f *fully represents* $\succeq$. It is easy to see that full representability is *strictly stronger* than partial representability.

It is well known (**?**) that a total preorder $\succeq$ is partially represented by some belief function Bel only if $\succeq$ satisfies (A_2). The following theorem is, therefore, an immediate corollary of Theorem 1.

Theorem 2. *(WADA$_{\succeq}$) entails that $\succeq$ is (extendible to a relation on $\mathcal{B}_n$ that is — fn. 7) partially represented by some belief function Bel.*

A natural question to ask at this point is: Does (WADA$_{\succeq}$) ensure that $\succeq$ is *fully* represented by some belief function Bel? The answer is *yes*. In order to see this, it helps to recognize that full representability by a belief function has a simple axiomatic characterization (**?**). Specifically, a total preorder $\succeq$ is fully represented by some belief function only if $\succeq$ satisfies (A_1), (A_2), and

(A_3) For all $p, q, r \in \mathcal{A}$, if p entails q and $\langle q, r \rangle$ are mutually exclusive, then:

$$q \succ p \Longrightarrow q \vee r \succ p \vee r.$$

Theorem 1 establishes that (WADA$_{\succeq}$) entails both (A_1) and (A_2). Moreover, it turns out that (WADA$_{\succeq}$) is also strong enough to entail (A_3). That is, we have the following theorem regarding (WADA$_{\succeq}$).

Theorem 3. *(WADA$_{\succeq}$) entails (A_3). [As a result, (WADA$_{\succeq}$) entails that $\succeq$ is (extendible to a relation on $\mathcal{B}_n$ that is — fn. 7) fully represented by some belief function (**?**).]*

Let's take stock. So far, we have encountered the following three coherence requirements for $\succeq$, in increasing order of strength, each of which is a consequence of our fundamental epistemic principle (WADA$_{\succeq}$).

($\mathfrak{C}_0$) $\succeq$ should be partially representable by some belief function Bel. This is equivalent to requiring that $\succeq$ (a total preorder) satisfies (A_2).

($\mathfrak{C}$) $\succeq$ should be fully representable by some plausibility measure Pl. This is equivalent to requiring that $\succeq$ (a total preorder) satisfies (A_1) and (A_2).

($\mathfrak{C}_1$) $\succeq$ should be fully representable by some belief function Bel. This is equivalent to requiring that $\succeq$ (a total preorder) satisfies (A_1), (A_2), and (A_3).

Moving beyond ($\mathfrak{C}_1$) takes us into the realm of *comparative probability*. A total preorder $\succeq$ is said to be a *comparative probability* relation only if $\succeq$ satisfies (A_1) and the following two additional axioms.

(A_4) For all $p \in \mathcal{A}$, $p \succeq \bot$.

(A_5) For all $p, q, r \in \mathcal{A}$, if $\langle p, q\rangle$ are mutually exclusive and $\langle p, r\rangle$ are mutually exclusive, then both of the following biconditionals are true:

$$q \succ r \iff p \vee q \succ p \vee r$$
$$\&$$
$$q \sim r \iff p \vee q \sim p \vee r$$

It is easy to show that {(A_1), (A_2)} jointly entail (A_4). So, $\succeq$ (a total preorder) is a comparative probability relation just in case $\succeq$ satisfies the three axioms (A_1), (A_2) and (A_5). Now, consider the following coherence requirement.

($\mathfrak{C}_2$) $\succeq$ should be be a comparative probability relation. This is equivalent to requiring that $\succeq$ (a total preorder) satisfies (A_1), (A_2) and (A_5).

It is well known (and not too difficult to prove) that (A_5) is *strictly stronger* than (A_3), in the presence of (A_1) and (A_2). Therefore, ($\mathfrak{C}_2$) is *strictly stronger* than ($\mathfrak{C}_1$). Moreover, in light of the following negative result, ($\mathfrak{C}_1$) is also where (WADA$_{\succeq}$)'s coherence implications peter out.

Theorem 4. *(WADA$_{\succeq}$) does not entail (A_5). In light of Theorems 1 and 3, this is equivalent to the claim that (WADA$_{\succeq}$) does not entail ($\mathfrak{C}_2$).*

Theorem 4 reveals that — just as in the case of full belief above — weak accuracy dominance avoidance yields a coherence requirement that is *weaker than (full) probabilistic representability* of the judgment set. Other analogies (and disanalogies) between the full belief case and the comparative confidence case will be discussed below (and in the Negative Phase). Meanwhile, we will examine a few more traditional coherence requirements for $\succeq$.

?? famously conjectured that all comparative probability relations are fully representable by some probability function. As it turns out, this conjecture is false. In fact, **?** showed that some comparative probability relations are *not even partially* representable by any probability function. As a result, the following coherence requirement for $\succeq$ is *strictly stronger* than ($\mathfrak{C}_2$).

($\mathfrak{C}_3$) $\succeq$ should be be partially representable by some probability function.

And, of course, ($\mathfrak{C}_3$) is *strictly weaker* than the following coherence requirement, which is the strongest of all the traditional coherence requirements.

($\mathfrak{C}_4$) $\succeq$ should be be fully representable by some probability function.

Moreover, it is easy to show (a) that ($\mathfrak{C}_3$) is *independent* of both ($\mathfrak{C}_2$) and ($\mathfrak{C}_1$), and (b) that ($\mathfrak{C}_3$) is *strictly stronger* than ($\mathfrak{C}$). The following axiomatic constraint is a slight weakening of (A_5).

($A_5^\star$) For all $p, q, r \in \mathcal{A}$, if $\langle p, q\rangle$ are mutually exclusive and $\langle p, r\rangle$ are mutually exclusive, then:

$$q \succ r \Longrightarrow p \vee r \not\succ p \vee q$$

And, the following coherence requirement is a (corresponding) weakening of ($\mathfrak{C}_2$).

($\mathfrak{C}_2^\star$) $\succeq$ should (be a total preorder and) satisfy (A_1), (A_2) and ($A_5^\star$).

Finally, the following negative result shows that ($\text{WADA}_\succeq$) is not strong enough to entail even *partial* probabilistic representability of $\succeq$.

Theorem 5. *($\text{WADA}_\succeq$) does not entail ($\mathfrak{C}_2^\star$). [And, as a result, ($\text{WADA}_\succeq$) does not entail ($\mathfrak{C}_3$) either.]*

The arrows in Figure 3 depict the logical relations between the traditional coherence requirements we've been discussing here. The superscripts on the coherence requirements in Figure 3 have the following meanings. If a coherence requirement is known to follow from ($\text{WADA}_\succeq$), then it gets a "✓". And, if a coherence requirement is known to follow from ($\text{SSADA}_\succeq$), then it gets a "✓✓." If a coherence requirement is known *not* to follow from ($\text{WADA}_\succeq$), then it gets an "✗".

All three "✓"s (and the "✓✓") in Figure 3 are established by our proof of Theorem 1. The "✗" on ($\mathfrak{C}_2$) is established by our proof of Theorem 4 and the "✗"s on ($\mathfrak{C}_2^\star$) and ($\mathfrak{C}_3$) are established by our proof of Theorem 5.

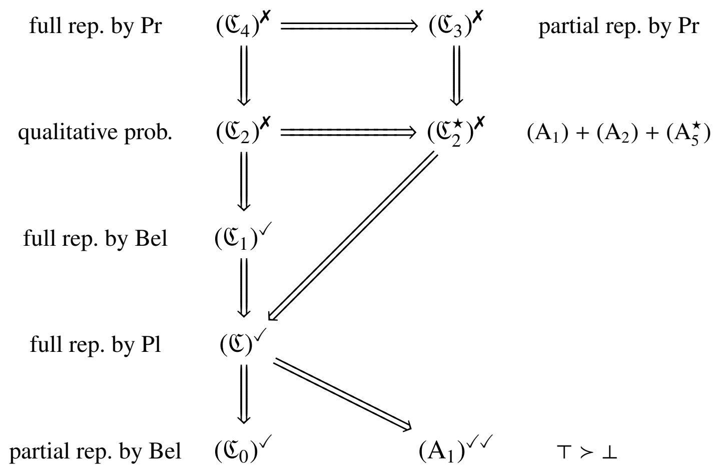

Figure 3: Logical relations between traditional $\succeq$-coherence requirements

5 Evidential Requirements for Comparative Confidence

So far, our approach has been based on *alethic* considerations. As in the case of full belief, alethic requirements do not seem to be the only kinds of epistemic requirements for comparative confidence. Indeed, we have already seen an example of a comparative confidence relation that seems to be epistemically reasonable/rational, but which *must* be *in*accurate. That example is depicted in Figure 1. It involves an agent who is indifferent between P and $\neg P$, where P is the claim that a coin which has been tossed has landed heads. Let's suppose, further, that the agent's total evidence entails that the coin in question is *fair* (and nothing else). Because P and $\neg P$ cannot have the same truth-value, our definition of inaccuracy entails that $P \sim \neg P$ *must* be inaccurate (*viz.*, alethically defective). Nonetheless, intuitively, the judgment $P \sim \neg P$ is *supported by the agent's total evidence*. It is for this reason that we deem the judgment in question to be epistemically rational/reasonable. At this point, we need to say something about the *general* evidential requirements for comparative confidence.

There are various proposals one could make regarding the (general) evidential requirements for comparative confidence. For the purposes of this investigation (and in keeping with the applications of our framework to full belief and numerical credence), we will adopt a *probabilistic* approach to evidential support.

In the case of full belief, we were working with an understanding of *probabilistic evidentialism* (recall its slogan "probabilities reflect evidence") that may be summarized *via* the following, general *necessary condition* for a (qualitative) judgment's being "supported by the total evidence".

($\mathcal{E}$) A qualitative judgment j (of type $\mathfrak{J}$) is supported by the total evidence *only if* there exists some probability function that probabilifies (*i.e.*, assigns probability greater than $1/2$ to) the vindicator (or "accuracy-maker") of j.

The vindicator of a belief that p [$B(p)$] is p and the vindicator of a disbelief that p [$D(p)$] is $\neg p$. The general principle ($\mathcal{E}$) thus led us to the following necessary condition for a belief set $\mathbf{B}$ (on an agenda $\mathcal{A}$) to be supported by the evidence.

> **Necessary Condition for Satisfying the Evidential Requirement for Full Belief**. A full belief set $\mathbf{B}$ satisfies the evidential requirement, *i.e.*, all judgments in $\mathbf{B}$ are supported by the total evidence, ***only*** *if*
>
> ($\mathcal{R}$) There exists some probability function that probabilifies, *i.e.*, assigns probability greater than $1/2$ to:
>
> (i) p, for each belief $B(p) \in \mathbf{B}$,
>
> (ii) $\neg p$, for each disbelief $D(p) \in \mathbf{B}$.

In the case of comparative confidence, we have a set of $\binom{m}{2}$ comparative confidence judgments over some m-proposition agenda $\mathcal{A}$. Some of these judgments may be strict comparisons $p \succ q$ and some may be indifferences $p \sim q$. Initially (*fn.* 8), it may seem natural to say that "the vindicator of" a strict comparison $p \succ q$ is $p \,\&\, \neg q$ and that "the vindicator of" an indifference $p \sim q$ is $p \equiv q$ (*fn.* 8). On this (*prima facie* natural) way of thinking (which, as we'll see shortly, is *flawed*), and in keeping with ($\mathcal{E}$), the corresponding (putative) necessary condition for satisfying the evidential requirement for comparative confidence would seem to be

> **Putative Necessary Condition for Satisfying the Evidential Requirement for Comparative Confidence**. A comparative confidence judgment set $\mathbf{C}$ satisfies the evidential requirement, *i.e.*, all judgments in the set are supported by the total evidence, ***only*** *if*
>
> There exists some probability function that probabilifies, *i.e.*, assigns probability greater than $1/2$ to:
>
> (i) $p \,\&\, \neg q$, for each strict judgment $p \succ q \in \mathbf{C}$,
>
> (ii) $p \equiv q$, for each indifference judgment $p \sim q \in \mathbf{C}$.

This *putative* evidential requirement is *not* a *bona fide* requirement. In fact, neither clause (i) nor clause (ii) expresses a genuine evidential requirement for comparative confidence relations. We have already seen why clause (ii) won't do. If we require that $p \equiv q$ be probabilified relative to some probability function, then this will rule out $P \sim \neg P$ as evidentially unsupportable, since $\Pr(P \equiv \neg P) = 0$, on *every* probability function. However, our fair coin toss example illustrates that the evidential requirements entail no such thing. The fair coin toss example also suffices to explain why (i) fails to express a genuine evidential requirement. In this case, we have $P \succ \bot$

and $\neg P \succ \bot$. And, intuitively, both of these judgments would be supported by the agent's total evidence. However, there can be no probability function that probabilifies the "vindicators" of both of these judgments simultaneously. That is, there can be no probability function that simultaneously assigns probability greater than $1/2$ to both P and $\neg P$. Therefore, clause (i) does not express a genuine (probabilistic) evidential requirement for comparative confidence either.

These considerations reveal an important disanalogy between full belief and comparative confidence. In the case of full belief, the "master (qualitative) probabilistic evidentialist" principle ($\mathcal{E}$) leads to a *bona fide* (probabilistic) evidential requirement [*viz.*, ($\mathcal{R}$)]. But, in the case of comparative confidence, an apparently analogous application of ($\mathcal{E}$) implies conditions that do *not* correspond to *bona fide* (probabilistic) evidential requirements. The problem here is that the application of ($\mathcal{E}$) to the case of individual comparative confidence judgments is *not* analogous (or appropriate). Strictly speaking, given our approach to Step 1 (*fn.* 8), there is no such thing as "the vindicator of" an individual comparative confidence judgment. There are (sometimes non-unique) vindicated *total comparative confidence ordering*(s). These are the ordering(s) which rank all the truths strictly above all the falsehoods (at the possible world in question). When it comes to *individual* comparative confidence judgments, all we can say is that there are two ways in which they can be *in*accurate — the two ways identified in Facts 1 and 2 above. We will return to these issues in the Negative Phase.

In any case, here is a natural alternative to the (putative, probabilistic) evidential requirement implied by ($\mathcal{E}$) in the case of comparative confidence.

> **Necessary Condition for Satisfying (Probabilistic) Evidential Requirements for Comparative Confidence**. A comparative confidence relation $\succeq$ (on an agenda $\mathcal{A}$) satisfies the (probabilistic) evidential requirement, *i.e.*, all of $\succeq$'s judgments are supported by the total evidence, ***only if***:

($\mathcal{R}_{\succeq}$) There exists some regular[17] probability function $\Pr(\cdot)$ such that:

(i) $\Pr(p) > \Pr(q)$, for each $p, q \in \mathcal{A}$ such that $p \succ q$,

(ii) $\Pr(p) = \Pr(q)$, for each $p, q \in \mathcal{A}$ such that $p \sim q$.

Of course, ($\mathcal{R}_{\succeq}$) entails the requirement that $\succeq$ be fully representable by some probability function, which was the strongest of the historical coherence requirements that we discussed above [*viz.*, ($\mathcal{R}_{\succeq}$) $\Rightarrow$ ($\mathfrak{C}_4$)]. The following theorem provides the key theoretical connection between alethic and evidential requirements for $\succeq$.

[17] A probability function is *regular* just in case it only assigns extreme probabilities to logically non-contingent claims. Equivalently, a probability function is regular only if it assigns non-zero probability to all logically possible worlds (**?**). Recall (*fn.* 4), we are assuming that our agents have Regular comparative confidence orderings. Moreover, the assumption of numerical regularity will be important to ensure the logical connection between ($\mathcal{R}_{\succeq}$) and (WADA$_{\succeq}$). We explain why numerical regularity is needed for these purposes in the proof of Theorem 6 in Appendix C, and we discuss some philosophical ramifications of numerical regularity in the Negative Phase.

Theorem 6. *If a comparative confidence relation $\succeq$ satisfies ($\mathcal{R}_{\succeq}$), then $\succeq$ satisfies (WADA$_{\succeq}$). That is, ($\mathcal{R}_{\succeq}$) entails (WADA$_{\succeq}$).*

This theorem is analogous to to our central Theorem **??** from Part I. And, the proof of Theorem 6 is analogous to the proof of Theorem **??**, insofar as it reveals (a) that any comparative confidence relation that is fully representable by some (regular) probability function Pr will also *minimize expected $\mathcal{I}_{\succeq}$-inaccuracy*, relative to its representing (evidential) probability function Pr; and, as a result, that (b) the measure $\mathcal{I}_{\succeq}$ is *evidentially proper*.[18] One final corollary of this central theorem for comparative confidence is that the entailment ($\mathcal{R}_{\succeq}$) $\Rightarrow$ (WADA$_{\succeq}$) will hold even if we assign different weights to different pairs of propositions in the agenda $\mathcal{A}$ (in our calculation of the total $\mathcal{I}_{\succeq}$-score of a comparative confidence judgment set). So, although our measure $\mathcal{I}_{\succeq}$ is additive, its use is compatible with assigning different "importance weights" to different pairs of propositions.

In the next section, we will step back and look at the big picture. We'll see how the alethic requirements and the evidential requirements for comparative confidence fit together, and how they dovetail with existing coherence requirements.

6 The Big Picture

From an alethic point of view, there are various norms which may govern comparative confidence judgments. In addition to (WADA$_{\succeq}$), there are also stronger alethic norms that one might consider. The strongest of these alethic norms would require that all comparative confidence judgments are (actually) *not in*accurate (in either of the ways identified in Facts 1 or 2).

> **Actual Vindication for Comparative Confidence Judgments** (AV$_{\succeq}$). No member of any set of comparative confidence judgments **C** should be *actually* inaccurate (*i.e.*, if $p \succ q \in \mathbf{C}$, then $p \vee \neg q$ should be *actually* true, and if $p \sim q \in \mathbf{C}$, then $p \equiv q$ should be *actually* true).

(AV$_{\succeq}$) is the analogue of the truth norm (TB) for full belief (which just is the requirement of actual vindication for full belief). A weaker alethic requirement would require only *possible* vindication for comparative confidence judgments.

> **Possible Vindication for Comparative Confidence Judgments** (PV$_{\succeq}$). For each set of comparative confidence judgments **C**, there should be some *possible* world at which no member of **C** is inaccurate (*i.e.*, there should be some world w such that (a) for each p, q such that $p \succ q \in \mathbf{C}$, $p \vee \neg q$ is true at w, and (b) for each p, q such that $p \sim q \in \mathbf{C}$, $p \equiv q$ is true at w).

[18] We will also prove (in Appendix C, along with our proof of Theorem 6) that the particular choice of inaccuracy scores (*i.e.*, the 2:1 ratio of scores in $\mathfrak{i}_{\succeq}$) for the two types of mistakes identified in Facts 1 and 2 above is *forced* by the requirement that the inaccuracy measure be evidentially proper.

($PV_{\geq}$) is the analogue of the deductive consistency requirement for full belief (which just is the requirement of possible vindication for full belief).

On the evidential side, we have the following (master) evidential requirement for comparative confidence judgments.

> **Evidential Requirement for Comparative Confidence Judgments** ($E_{\geq}$). For each set of comparative confidence judgments **C**, all of **C**'s members should be supported by the total evidence.

From a *probabilistic evidentialist* perspective, ($E_{\geq}$) requires (at least[19]) that $\geq$ satisfy condition ($\mathcal{R}_{\geq}$) above. At this point, we have all the ingredients we need to reveal the big picture of norms and requirements discussed in this part of the book.

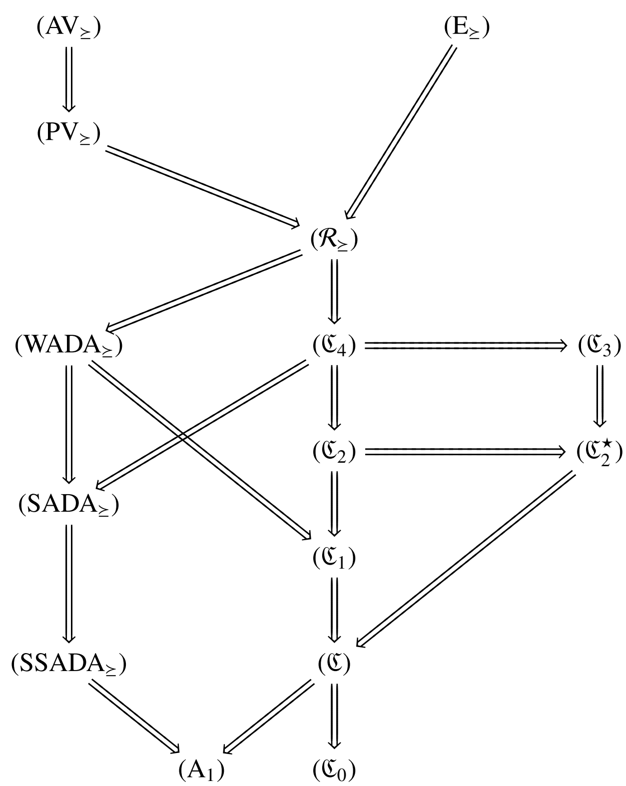

Figure 4: Logical relations between all norms and requirements for comparative confidence

[19]Intuitively, just as in the case of full belief, the evidential requirement ($E_{\geq}$) will be *stronger* than ($\mathcal{R}_{\geq}$), since it will require representability by a *specific* (regular) *evidential* probability function, which gauges the evidential support relations in the context in which the $\geq$-judgments are formed.

Figure 4 reveals that (as in the case of full belief) our requirement ($\mathcal{R}_{\succeq}$) plays a central role in our story. For ($\mathcal{R}_{\succeq}$) is entailed by both of the strong alethic requirements ($AV_{\succeq}$) and ($PV_{\succeq}$), and the (master) evidential requirement ($E_{\succeq}$).[20] Of course, ($\mathcal{R}_{\succeq}$) entails all of the traditional coherence requirements ($\mathfrak{C}_i$) discussed above. Moreover, as the proof of Theorem 6 reveals, ($\mathcal{R}_{\succeq}$) also entails ($WADA_{\succeq}$).

Finally, our arguments above reveal that ($WADA_{\succeq}$) implies that an agent's comparative confidence ordering should be (extendible to a relation on $\mathcal{B}_n$ that is — *fn.* 7) fully representable by some belief function [($\mathfrak{C}_1$)]. This provides an epistemic justification for the "coherence" of comparative confidence judgments in the sense of Dempster-Shafer (**???**). Adherents of representability by a belief function as a requirement of rationality should welcome this result — especially, since we know of no existing epistemic justification for ($\mathfrak{C}_1$) as a formal coherence requirement for comparative confidence.

Comparative Confidence: Negative Phase

What?

Richard Milhous Nixon

7 Why Not the Stronger Notion of "Accuracy" for Individual $\succeq$ Judgments?

Recall the assumption (which we discussed above — *fn.* 8) that there is a *unique* vindicated ordering $\mathring{\succeq}_w$ in each world w, which is defined as follows

$$p \mathring{\succeq}_w q \begin{cases} p \mathring{\succ}_w q & \text{if } p \text{ is true in } w \text{ and } q \text{ is false in } w, \\ p \mathring{\sim}_w q & \text{if } p \text{ and } q \text{ have the same truth-value in } w. \end{cases}$$

This assumption leads to a weaker notion of "*in*accuracy of a comparative confidence ordering $\succeq$" — *non-identity* of $\succeq$ and $\mathring{\succeq}_w$ — than our (official) definition, which is compatible with the existence of *many* "vindicated orderings" in a given world w — so long as they each rank all the truths strictly above all the falsehoods at w. This stronger notion of "accuracy" suggests other possibilities for non-zero inaccuracy scores that should be assigned by our point-wise measure $\mathfrak{i}_{\succeq}$. Figure 1 depicts all the possible point-wise scoring schemes for individual comparative confidence judgments. Here, we stipulate (as a matter of convention) that the inaccuracy score for an indifference judgment regarding propositions having different truth values is 1. The zeros in the table are uncontroversial, assuming that our measure is $\mathfrak{i}$ is going to be *truth-directed*

[20]Because all of our comparative confidence relations are *regular* total preorders, we should always be able to construct a regular probability function $\Pr(\cdot)$ which represents a comparative confidence $\succeq$ relation that is "possibly vindicated" in the sense of ($PV_{\succeq}$) in possible world w. I haven't been able to specify a general algorithm for such a construction, but I'm confident that there must be such a construction. Assistance welcome!

w_i	p	q	$p \succ q$	$p \sim q$
w_1	T	T	b	0
w_2	T	F	0	1
w_3	F	T	a	1
w_4	F	F	b	0

Table 1: All possible point-wise scoring schemes for individual comparative confidence judgments.

(*fn.* **??**). The only two parameters that remain are a and b. Our (stronger) notion of inaccuracy for individual comparative confidence judgments — as embodied in our definition of $i_{\succeq}$ — implies that $b = 0$. That's because these judgments do not (in and of themselves) entail that the relation $\succeq$ is inaccurate in our official sense. However, these judgments *would* imply that $\succeq$ is inaccurate in the (weaker) sense above (*viz.*, *being non-identical with* $\mathring{\succeq}_w$). Unfortunately, *no point-wise measure of inaccuracy that assigns anything other than* $a := 2$ *and* $b := 0$ *leads to an evidentially proper measure of total inaccuracy.*[21] For this reason, we *must* assign $b := 0$ if we want an evidentially proper measure of inaccuracy to emerge from our approach. This is the main reason we have opted for our weaker notion of "accuracy" (and our stronger notion of "inaccuracy").[22] In the next section, we examine a formal analogy between full belief and comparative confidence that has been explored in the literature.

8 A Formal Analogy Between Full Belief and Comparative Confidence: Some Cautionary Remarks

It is sometimes argued (**?**) that it is possible to construct an adequate measure $\delta(\succeq, w)$ of the inaccuracy of a comparative confidence relation $\succeq$ by (a) "translating" its comparative confidence judgments $p \succeq q$ into *full beliefs* that some (extensional) "is at least as plausible as" relation obtains between p and q, and (b) applying one's favored measure of inaccuracy for belief sets to these "translated" comparative confidence judgment sets. Specifically, we can use our (evidentially proper) measure of the inaccuracy of a belief set $\mathcal{I}(\mathbf{B}, w)$ to construct a measure $\delta(\succeq, w)$ of the inaccuracy of comparative confidence relations.

Here's how the construction works. For each pair of propositions $p, q \in \mathcal{A}$, our

[21]This is proven in Appendix C (after the proof of Theorem 6).

[22]We conjecture that it is possible to think of our measure $\mathcal{I}_{\succeq}$ as a measure of "distance from vindication", even though there may be *many* "vindicated relations" $\{\mathring{\succeq}_w\}$ in a given possible world w. The idea would be to define $\succeq$'s "distance from vindication at w" as the *edit distance* (Deza & Deza **?**) between $\succeq$ and the member of $\{\mathring{\succeq}_w\}$ that is closest to $\succeq$ (in edit distance). We thank Ben Levinstein for suggesting this conjecture. This is another interesting open theoretical question.

agents make exactly one comparative judgment out of the following three

$$\{p \succ q, q \succ p, p \sim q\}.$$

If we define the relation:

$$p \succcurlyeq q \;\; (p \succ q) \text{ or } (p \sim q),$$

then these three judgments correspond to the following three $\succcurlyeq$-judgment *pairs*

$$\{\langle p \succcurlyeq q, q \not\succcurlyeq p\rangle, \langle p \not\succcurlyeq q, q \succcurlyeq p\rangle, \langle p \succcurlyeq q, q \succcurlyeq p\rangle\}.$$

Next, we can "translate" these $\succcurlyeq$-pairs into pairs of *beliefs* and *disbeliefs*, as follows

$$\{\langle B(Rpq), D(Rqp)\rangle, \langle D(Rpq), B(Rqp)\rangle, \langle B(Rpq), B(Rqp)\rangle\},$$

where Rpq is interpreted as some (extensional) "at least as plausible as" relation that may obtain between propositions p and q. To complete the analogy, we just need an extensional (*viz.*, truth-functional) interpretation of the relation R. Given our definition of inaccuracy for comparative confidence judgments $p \succeq q$, the appropriate (extensional) interpretation of $\ulcorner Rpq \urcorner$ is $\ulcorner p \vee \neg q \urcorner$.

We can now construct a measure of inaccuracy δ for (sets of) comparative confidence judgments in terms of our naïve mistake-counting measure of inaccuracy $\mathcal{I}$ for belief sets, *via* the above translation. First, for each comparative confidence judgment in a set $\mathbf{C}$, we translate it into its corresponding belief/disbelief pair. Then, we combine all these pairs into one big belief set $\mathbf{B_C}$. In this way, every comparative confidence judgment set $\mathbf{C}$ will have a (unique) corresponding belief set $\mathbf{B_C}$. Finally, applying our naïve mistake-counting measure of inaccuracy for belief sets to $\mathbf{B_C}$ yields $\mathcal{I}(\mathbf{B_C}, w)$. It is easy to show (**?**) that this procedure yields an inaccuracy measure $\delta(\mathbf{C}, w) \;\; \mathcal{I}(\mathbf{B_C}, w)$ for comparative confidence judgment sets, which is equivalent to the measure that simply *counts the number of differences between the adjacency matrix of* $\mathbf{C}$ *and the adjacency matrix of "the unique vindicated* $\succeq$*-relation at* w*"* ($\mathring{\succeq}_w$) that we discussed in the previous section (*fn.* 8). In other words, $\delta(\mathbf{C}, w)$ is the *Kemeny distance* (Deza & Deza **?**) between the two total preorders $\succeq$ (*viz.*, $\mathbf{C}$) and $\mathring{\succeq}_w$.

Because the Kemeny measure — like our simple, mistake-counting measure Δ above — simply *counts* "the number of mistakes" in a comparative judgment set at a world, it *scores all mistakes equally*. As a result, we have the following negative result regarding the Kemeny measure of inaccuracy of a $\succeq$-relation.

Theorem 7. *The entailment ($\mathcal{R}_{\succeq}$) $\Rightarrow$ (WADA$_{\succeq}$) does not hold, if we use the Kemeny measure of inaccuracy (δ) rather than our (evidentially proper) measure $\mathcal{I}_{\succeq}$. And, as a result, the Kemeny measure of inaccuracy is not evidentially proper (even for regular evidential probability functions).*

So, like Δ, the Kemeny measure runs afoul of our evidential requirements for comparative confidence. This is interesting, as it shows that the standard way of defining the inaccuracy of a comparative confidence relation, *via* "translation" into a corresponding full belief set *fails to preserve the property of evidential propriety* that is enjoyed by our measure of inaccuracy $\mathcal{I}$ for full belief sets.

The proof of Theorem 7 (Appendix C) requires examples involving algebras containing at least three states (or worlds). But, there is a simpler way to see that this formal analogy between comparative confidence and full belief can (if taken too literally) yield incorrect conclusions about evidential requirements. Consider the simple agenda $\mathcal{A} = \{P, \neg P\}$. And, consider the comparative confidence relation on $\mathcal{A}$ such that $P \sim \neg P$. This corresponds to the singleton comparative confidence judgment set $\mathbf{C} = \{P \sim \neg P\}$. Now, if we "translate" $\mathbf{C}$ into its corresponding belief set, we get: $\mathbf{B}_\mathbf{C} = \{B(P \vee P), B(\neg P \vee \neg P)\}$, which, by logical omniscience, is just the contradictory pair of beliefs $\mathbf{B}_\mathbf{C} = \{B(P), B(\neg P)\}$. Neither $\mathbf{C}$ nor $\mathbf{B}_\mathbf{C}$ can be vindicated (in any possible world). In this (formal, alethic) sense, the analogy between $\mathbf{B}_\mathbf{C}$ and $\mathbf{C}$ is tight. But, as we discussed above, because it violates ($\mathcal{R}$), the pair of beliefs in $\mathbf{B}_\mathbf{C}$ *cannot both be supported by any body of evidence*. However, the original comparative confidence judgment $P \sim \neg P$ comprising $\mathbf{C}$ *can* be supported by some bodies of evidence (*e.g.*, our fair coin case above). The moral of this story is that we must not take formal (alethic) analogies between comparative confidence and full belief too seriously, since this will inevitably lead to inappropriate (analogical) epistemic conclusions regarding evidential requirements.[23]

9 Regularity and Infinite Underlying Possibility Spaces

In the case of full belief, the (naïve, mistake-counting) measure of doxastic inaccuracy we discussed only makes sense as applied to finite agendas. However, our probabilistic coherence requirements ($\mathcal{R}_r$) can be non-trivially applied to infinite agendas. Of course, there are many controversies involving probability models over infinite structures (especially, uncountable ones — see below). But, in the full belief case, we can largely set those aside, and apply our favorite account of numerical (evidential) probability (for infinite epistemic possibility spaces) to give substance to our requirements ($\mathcal{R}_r$), as applied to infinite agendas.

In the case of comparative confidence, however, there are some additional wrinkles that are worth discussing here. Generally, it seems rationally permissible to rank a contingent claim P (which is not known to be false) *strictly above* a contradiction in one's comparative confidence ordering (indeed, we are assuming comparative Regularity, which *entails* this). But, some contingent claims involving an uncountable set of (underlying, epistemic) possibilities also seem to have a numerical (evidential)

[23]Before settling on the present way of modeling comparative confidence judgments and their coherence requirements, we attempted to model comparative confidence judgments *as* beliefs about comparative plausibility relations. That approach ultimately failed to yield plausible results, and was plagued by puzzles and paradoxes (not unrelated to the cautionary remarks of this section). We are indebted to David Christensen, Ben Levinstein and Daniel Berntson for pressing on these disanalogies and nudging us toward our new approach. See (**?**, Chapter 1) for further discussion.

probability of (identically) *zero*. This combination is troublesome, given our present approach. To make things concrete, let P be the claim that a fair coin will repeatedly land heads *infinitely often* (if tossed an infinite number of times).[24] We are inclined to agree with recent authors (**???**) who argue for the following two claims.[25]

(1) It is rationally permissible to have a comparative confidence ordering $\succeq$ such that $P \succ \bot$ (indeed, we're assuming $\succeq$-*regularity*, which *entails* this).

(2) All (evidential) probability functions are such that $\Pr(P) = \Pr(\bot) = 0$.

Unfortunately, (1) and (2) jointly *contradict* the claim that $(\mathcal{R}_{\succeq})$ is a requirement of epistemic rationality. That is, the idea that "(numerical) probabilities reflect evidence" seems to run into trouble when applied to examples such as these.

Such examples are very subtle, and a proper discussion of them is beyond our present scope. However, I will make a few brief remarks about such cases. First, the agenda in question is actually *finite* — it involves just two propositions: $\{P, \bot\}$. So, we can apply our naïve framework directly to such examples, and what we will find is that, in general, $P \succ \bot$ will be *non-dominated* in accuracy. So, in support of (1), at least we can say that $P \succ \bot$ doesn't violate ($\text{WADA}_{\succeq}$). Hence, the trouble here seems to lie squarely with claim (2). It is only when we accept (2) that these examples cause problems for our overall approach, which places $(\mathcal{R}_{\succeq})$ at center stage. In other words, these cases put pressure on the claim that $(\mathcal{R}_{\succeq})$ is a (universal) requirement of epistemic rationality. In light of these examples, we are inclined to restrict the domain of application of the present models [*viz.*, $(\mathcal{R}_{\succeq})$] to cases involving finite underlying epistemic possibility spaces.[26]

Another important upshot of these sorts of examples is that they call into question the common assumption that comparative confidence *reduces to* numerical credence (**?**). Indeed, these kinds of examples seem to suggest that comparative confidence cannot even *supervene* on numerical credence (at least, not in the usual way, *via* full probabilistic representability of the comparative confidence ordering by the agent's numerical credence function). Such a failure of supervenience of comparative confidence on numerical credence would support a *pluralistic* stance regarding these two kinds of attitudes. While we take no (official) stand on this issue here, we are (as we mentioned above) inclined toward a pluralistic stance (quite generally) anyhow. So this result would not be unwelcome.

[24]If you don't like this example, we could let P be the claim that an infinitely thin dart, which is thrown at the real unit interval, lands *exactly* on (say) the point $1/2$ (**?**).

[25]Strictly speaking, most authors argue for a *credal* rendition of claim (2). That is, most authors focus on whether claim (2) is true for *epistemically rational credences*. Some authors have voiced similar reservations regarding *chances* (**?**). I suppose it is *possible* that *evidential* probabilities could violate (2), even if rational credences (and chances) do not. But, this seems implausible.

[26]Alternatively, we could plump for an approach to epistemic decision theory that does not rely so heavily on numerical representation theorems, and places more emphasis on dominance-style reasoning — especially in cases involving uncountable underlying epistemic possibility spaces (**?**). Note: we are not optimistic about the prospects of using non-standard analysis to handle such problems. See (**???**) for some pessimistic arguments.

10 The Ordering Assumptions (Opinionation and Transitivity) Revisited

For the purposes of the Positive Phase, we presented a simplified version of the framework in which $\geq$ is assumed to be a total preorder. However, three of our ordering assumptions ($\geq$-opinionation, $\succ$-transitivity and $\sim$-transitivity) have been the source of considerable controversy in the literature (**???**).

Our assumption of $\geq$-opinionation as a constraint on $\geq$ is analogous to our assumption of opinionation in the case of full belief. For simplicity, this monograph is (almost entirely) concerned with opinionated judgment sets. As a result, I will not discuss $\geq$-opinionation further, since that would get us into a more general discussion of suspension-like attitudes, which is beyond our present scope.

The more interesting ordering assumptions (for present purposes) are our two transitivity assumptions. Interestingly, *neither* of our transitivity assumptions follows from ($\text{WADA}_{\geq}$). Indeed, we can say something even stronger than this. Neither of the following two forms of transitivity is implied by ($\text{WADA}_{\geq}$).

> **Weak $\succ$-Transitivity**. For all $p, q, r \in \mathcal{A}$, *if* $p \succ q$ and $q \succ r$, *then* $r \not\succ p$.

> **Transitivity of** $\sim$. For all $p, q, r \in \mathcal{A}$, *if* $p \sim q$ and $q \sim r$, *then* $p \sim r$.

Weak $\succ$-Transitivity is sometimes called *weak consistency*, and it seems to be the least controversial of the varieties of $\geq$-transitivity. Transitivity of $\sim$ seems to be the most controversial of the varieties of $\geq$-transitivity (**?**). Here is a potential counterexample to the transitivity of epistemic indifference.

> S observes a bank robbery. S gets a good look at the robber (r_0), who has a full head of hair. The police create n perfect duplicates of the robber (r_i). They remove i hairs from the head of r_i, and make a *line-up*: $r_0, r_1, r_2, r_3, \ldots, r_N$. They show S *only neighboring pairs*: $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$, $\langle r_2, r_3\rangle, \ldots, \langle r_{N-1}, r_N\rangle$. Now, let p_i r_i is the robber (*i.e.*, $r_i = r_0$). On the basis of the visual evidence S obtains from observing the pairs, S comes form the pairwise indifference judgments
> $p_i \sim p_{i+1}$, for each i. But, at the end of the day, the police show S *one last pair*: $\langle r_0, r_N\rangle$, where r_N has *zero* hairs on his head. Of course, S judges $p_0 \succ p_N$.

In light of such examples, the fact that transitivity of indifference does not have an accuracy-dominance rationale should (perhaps) not be terribly surprising. But, it is interesting that even the least controversial variety of transitivity (weak $\succ$-transitivity) fails to have an accuracy dominance rationale.[27]

[27] In the context of numerical credence, the ordering assumptions are simply *baked-in via* the presupposition that numerical credence functions are *real-valued* (and that the relation $\geq$ on the real numbers is — *a fortiori* — a total order). So, we don't view the fact that we cannot offer a (qualitative) rationale for the ordering assumptions as a major shortcoming of our approach. In this sense, it seems that — from an epistemic point of view — assumptions like transitivity must be *built-in* from the outset (perhaps, as constitutive aspects of concepts describable *via* an "at least as..." locution).

The present discussion raises a crucial theoretical question: *Precisely which* properties of $\succeq$ follow from (WADA$_{\succeq}$)? Ideally, it would be nice to have an (sound and complete) *axiomatization* of the formal properties of $\succeq$ that are consequences of (WADA$_{\succeq}$). This remains an important unsolved theoretical problem.

11 The "Circularity Worry" About ($\mathcal{R}_{\succeq}$) and Our Philosophical Methodology

One might worry that adopting ($\mathcal{R}_{\succeq}$) as an evidential requirement for comparative confidence is somehow "circular". If our aim is to give an argument for *probabilism* as an epistemic coherence requirement for comparative confidence relations, then it would seem that assuming ($\mathcal{R}_{\succeq}$) is *question-begging*.

The aim of the present monograph is not so much to provide "arguments for" particular coherence requirements (in the usual sense). Rather, the aim is to put forward a simple and strong *package* of epistemic principles, and to explain how coherence requirements — for various types of judgments — can be seen as emerging from this package in a unified way. Our package has three main components.

(P_1) *Alethic* (or *Accuracy*) norms (and requirements). These are *extensional* (*viz.*, determined by the *truth-values* of the contents of the judgments in question) norms (and requirements). *E.g.*, (TB), (CB), (AV$_{\succeq}$), (PV$_{\succeq}$).

(P_2) *Evidential* norms (and requirements). These are *probabilistic* (*viz.*, suitable varieties of *probabilistic representability*). *E.g.*, ($\mathcal{R}$) and ($\mathcal{R}_{\succeq}$).

(P_3) *Coherence requirements*, which include (a) *minimization of expected inaccuracy* (relative to some evidential probability function Pr) principles, and (b) *accuracy-dominance avoidance* principles which are entailed by them. In all of our applications, the accuracy dominance avoidance principles [*e.g.*, (WADA) and (WADA$_{\succeq}$)] are offered as coherence requirements. And, in all applications of our framework, *evidential propriety* of the inaccuracy measure is essential for grounding these requirements. In the case of full belief, (P_3) is established by the proof of Theorem **??** (see Appendix B), and in the case of comparative confidence, (P_3) is established by the proof of Theorem 6 (see Appendix C).

You may have noticed that both of our "big picture" figures of norms and requirements so far (*i.e.*, Figures **??** and 4) share a common "tree" structure. Abstracting away from the details, we can now see that this "tree" always has (P_1) and (P_2) on its branches, and (P_3) on its its trunk (and the trunk of this "tree" is also where the epistemic rational requirements reside). Figure 5 gives a generic "Big Picture" representation of the logical structure of (P_1)–(P_3). Part III will have a similarly "tree" shaped "big picture" diagram (of norms and requirements for credence).

From this perspective, it is a *basic assumption* of this monograph [(P_2)] that "(numerical) probabilities reflect evidence." This is not something that we are aiming to

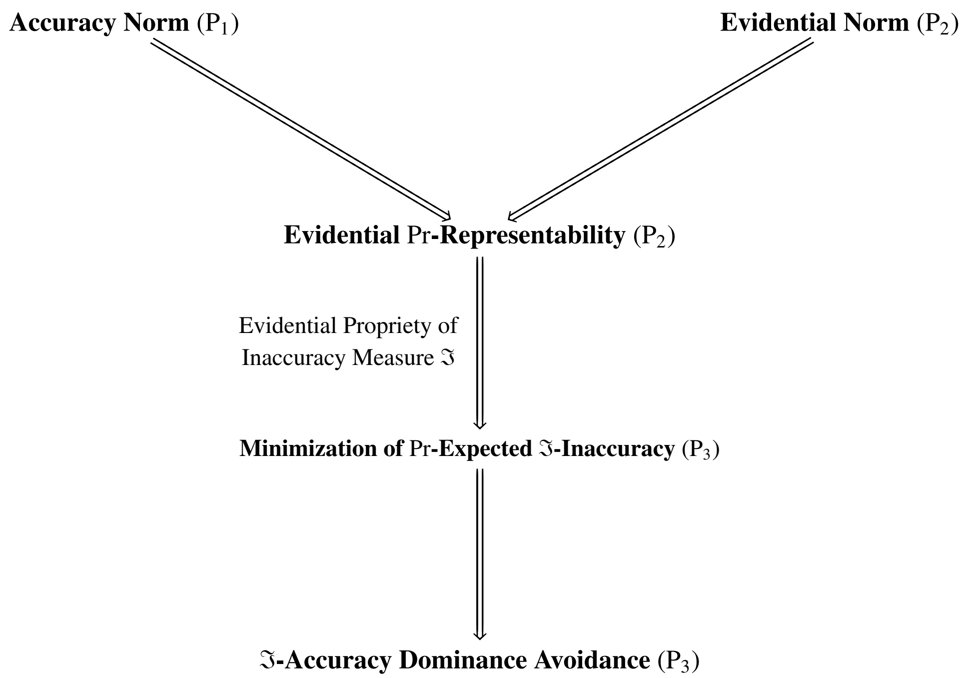

Figure 5: Generic "Big Picture" Diagram of our Package (P_1)–(P_3)

provide some sort of (direct or independent) "argument for" (in the traditional "arguments for probabilism" sense). Although, we will discuss this possibility in Part III below. Perhaps a better way to view our present methodology is in the spirit of a "best explication" (**?**, Chapter 1) or a "best systems analysis" (**??**) of formal epistemic coherence requirements.

We maintain that (P_1)–(P_3) can undergird the best *explication* of (or a "best systems analysis" of) formal epistemic coherence. Our package has many theoretical virtues, including simplicity, strength, fit, *etc.* And, we think that the quality of the package *as a whole* serves to (indirectly) lend some credence to each of its parts.

If someone were to offer a *better* explication of formal epistemic coherence (or a "better system" of requirements of epistemic rationality), then that would pose a problem for us. But, "circularity" (as such) is not a pressing worry for us. We'll be satisfied if we manage to systematize the salient body of facts (*i.e.*, what we take to be the salient data) regarding epistemic rationality in an explanatory and illuminating way. This methodological stance will play an even more prominent role in the final part of the book (on numerical credence), to which we now turn.

Appendix: Proofs of Theorems

PROOF OF THEOREM 1
[(WADA$_{\succeq}$) $\Rightarrow$ ($\mathfrak{C}$)]

Suppose $\succeq$ violates ($\mathfrak{C}$). There are only two ways this can happen. Either $\succeq$ violates (A_1) or $\succeq$ violates (A_2).

Suppose $\succeq$ violates (A_1). Because $\succeq$ is total, this means $\succeq$ is such that $\bot \succeq \top$. Consider the relation $\succeq'$ which agrees with $\succeq$ on all comparisons outside the $\langle \bot, \top \rangle$-fragment, but which is such that $\top \succ' \bot$. Because $\top$ & $\neg\bot$ *is true in every possible world* w, it follows that $\mathbf{M}(\succeq', w) \subset \mathbf{M}(\succeq, w)$ *in every possible world* w. Thus, (A_1) actually follows from (SSADA$_{\succeq}$). And, of course, (WADA$_{\succeq}$) entails (SSADA$_{\succeq}$).

Suppose $\succeq$ violates (A_2). Because $\succeq$ is total, this means there is a pair of propositions p and q in $\mathcal{A}$ such that (a) p entails q but (b) $p \succ q$. Consider the relation $\succeq'$ which agrees with $\succeq$ on every judgment *except* (b), and which is such that $q \succ' p$. Table 2 depicts the $\langle p, q \rangle$-fragment of the relations $\succeq$ and $\succeq'$ in the three salient possible worlds (the second row/world is *impossible*, since p entails q). By (b) and (LO), p and q are not logically equivalent. Thus, world w_2 is a live possibility. Therefore, $\succeq'$ weakly $\mathcal{I}_{\succeq}$-dominates $\succeq$.

w_i	p	q	$\succeq$	$\succeq'$	$\mathcal{I}_{\succeq}(\succeq, w_i)$	$\mathcal{I}_{\succeq}(\succeq', w_i)$
w_1	T	T	$p \succ q$	$q \succ' p$	0	0
	T	F				
w_2	F	T	$p \succ q$	$q \succ' p$	2	0
w_3	F	F	$p \succ q$	$q \succ' p$	0	0

Table 2: The $\langle p, q \rangle$-fragments of $\succeq$ and $\succeq'$ in the 3 salient possible worlds [(A_2) case of Theorem 1].

VERIFYING THEOREM 3
[(WADA$_{\succeq}$) $\Rightarrow$ (A_3)]

Suppose $\succeq$ violates (A_3). Because $\succeq$ is total, this means there must exist $p, q, r \in \mathcal{A}$ such that (a) $p \vDash q$, (b) $\langle q, r \rangle$ are mutually exclusive, (c) $q \succ p$, but (d) $p \vee r \succeq q \vee r$. Let $\succeq'$ agree with $\succeq$ on every judgment, *except* (d). That is, let $\succeq'$ be such that (e) $q \succ' p$ and (f) $q \vee r \succ' p \vee r$. There are only four worlds compatible with the precondition of (A_3), and these are depicted in Table 3, below.

By (c) and (LO), p and q are not logically equivalent. Moreover, it is easy to verify that (f) will *not* be inaccurate in *any* of these four worlds, while (d) *must be inaccurate in world* w_2 (see the companion *Mathematica* notebook for these calculations). This suffices to show that $\succeq'$ *weakly* $\mathcal{I}_{\succeq}$-dominates $\succeq$.

VERIFYING THEOREM 4 AND THEOREM 5
[(WADA$_{\succeq}$) $\not\Rightarrow$ (A_5) and (WADA$_{\succeq}$) $\not\Rightarrow$ ($A_5^{\star}$)]

p	q	r	possible world w_i
T	**T**	**T**	
T	**T**	**F**	w_1
T	**F**	**T**	
T	**F**	**F**	
F	**T**	**T**	
F	**T**	**F**	w_2
F	**F**	**T**	w_3
F	**F**	**F**	w_4

Table 3: Schematic truth-table depicting the four (4) possible worlds compatible with the precondition of (A_3).

Suppose (a) $\langle p, q\rangle$ and $\langle p, r\rangle$ are mutually exclusive, (b) $q \succ r$, and (c) $p \vee r \succ p \vee q$. It can be shown that *there is no binary relation* $\succeq'$ on the agenda $\langle p, q, r\rangle$ such that (i) $\succeq'$ agrees with $\succeq$ on all judgments *except* (b) and (c), and (ii) $\succeq'$ weakly $\mathcal{I}_{\succeq}$-dominates $\succeq$. There are only four alternative judgment sets that need to be compared with $\{(b), (c)\}$, in terms of their $\mathcal{I}_{\succeq}$-values across the five possible worlds (w_1–w_5) compatible with the precondition of (A_5), which are depicted in Table 4. (1) $\{q \sim r, p \vee r \succ p \vee q\}$, (2) $\{r \succ q, p \vee r \succ p \vee q\}$, (3) $\{q \succ r, p \vee r \sim p \vee q\}$, and (4) $\{q \sim r, p \vee r \sim p \vee q\}$. It is easy to verify that none of these alternative judgment sets weakly $\mathcal{I}_{\succeq}$-dominates the set $\{(b), (c)\}$, across the five salient possible worlds (see the companion *Mathematica* notebook for these calculations). Note: this argument establishes the *stronger* claim (Theorem 5) that (WADA) does *not* entail ($A_5^{\star}$).

p	q	r	possible world w_i
T	**T**	**T**	
T	**T**	**F**	
T	**F**	**T**	
T	**F**	**F**	w_1
F	**T**	**T**	w_2
F	**T**	**F**	w_3
F	**F**	**T**	w_4
F	**F**	**F**	w_5

Table 4: Schematic truth-table depicting the five (5) possible worlds compatible with the precondition of (A_5) [and ($A_5^{\star}$)].

Proof of Theorem 6
$[(\mathcal{R}_{\succeq}) \Rightarrow (\text{WADA}_{\succeq})$ and $(\mathfrak{C}_4) \Rightarrow (\text{SADA}_{\succeq})]$

We will first show that $(\mathfrak{C}_4) \Rightarrow (\text{SADA}_{\succeq})$ and then we'll prove $(\mathcal{R}_{\succeq}) \Rightarrow (\text{WADA}_{\succeq})$. Suppose $(\mathfrak{C}_4)$ holds. That is, suppose $\Pr(\cdot)$ is a probability function that fully represents $\succeq$ (on agenda $\mathcal{A}$). Consider the expected $\mathcal{I}_{\succeq}$-inaccuracy, as calculated by $\Pr(\cdot)$, of $\succeq$. This is given by the sum over all possible worlds w of $\Pr(w) \cdot \mathcal{I}_{\succeq}(\succeq, w)$. Since $\mathcal{I}_{\succeq}(\succeq, w)$ is a sum of the components $\mathfrak{i}_{\succeq}(p \succeq q, w)$ for each pair of propositions $p, q \in \mathcal{A}$, and since expectations are linear, the expected inaccuracy is the sum of the expectation of these components. The expectation of the component $\mathfrak{i}_{\succeq}(p \succ q, w)$ is $2 \cdot \Pr(q \,\&\, \neg p)$ while the expectation of the component $\mathfrak{i}_{\succeq}(p \sim q, w)$ is $\Pr(p \not\equiv p)$. Now, there are two possibilities for a given pair $p, q \in \mathcal{A}$.

1. $\Pr(p) > \Pr(q)$. In this case, the expected inaccuracy of $p \succ q$ is $2 \cdot \Pr(q \,\&\, \neg p)$, which is *strictly less than* the expected inaccuracy of either $p \sim q$, which is given by $\Pr(p \not\equiv p)$ or $q \succ p$, which is given by $2 \cdot \Pr(p \,\&\, \neg q)$. So, if $\Pr(p) > \Pr(q)$, then $p \succ q$ *uniquely* minimizes expected $\mathfrak{i}_{\succeq}$-inaccuracy.

2. $\Pr(p) = \Pr(q)$. In this case, the expected inaccuracy of $p \sim q$ is $\Pr(p \not\equiv p)$, which *is equal to* the expected inaccuracy of $p \succ q$, which is given by $2 \cdot \Pr(\neg p \,\&\, q)$. So, if $\Pr(p) = \Pr(q)$, then $p \sim q$ non-uniquely minimizes expected $\mathfrak{i}_{\succeq}$-inaccuracy, *i.e.*, all judgments have the same $\mathfrak{i}_{\succeq}$-expectation.

Thus, since $\Pr(\cdot)$ fully represents $\succeq$, this means that, $\succeq$ will have an expected $\mathcal{I}_{\succeq}$-inaccuracy that is *no greater than* that of any other $\succeq$-relation (on $\mathcal{A}$) with respect to $\Pr(\cdot)$. Thus, the measure $\mathcal{I}_{\succeq}$ is *evidentially proper*. And, no relation $\succeq'$ can *strictly* dominate $\succeq$ in $\mathcal{I}_{\succeq}$-inaccuracy (on $\mathcal{A}$), since such a $\succeq'$ would have a *strictly lower* Pr-expected $\mathcal{I}_{\succeq}$-inaccuracy than $\succeq$ (on $\mathcal{A}$). In other words, $(\mathfrak{C}_4)$ entails $(\text{SADA}_{\succeq})$. Now, suppose that $(\mathcal{R}_{\succeq})$ holds. That is, suppose $\Pr(\cdot)$ is a *regular* probability function that fully represents $\succeq$ (on $\mathcal{A}$). Then, no relation $\succeq'$ can *weakly* dominate $\succeq$ in $\mathcal{I}_{\succeq}$-inaccuracy (on $\mathcal{A}$), since such a relation would have a *strictly lower* Pr-expected $\mathcal{I}_{\succeq}$-inaccuracy than $\succeq$ (on $\mathcal{A}$). Therefore, $(\mathcal{R}_{\succeq})$ entails $(\text{WADA}_{\succeq})$.

Verifying The Uniqueness of Our Evidentially Proper Measure $\mathfrak{i}_{\succeq}$

Theorem. *$a := 2$ and $b := 0$ is the only numerical assignment to a and b which ensures that the following parametric definition of $\mathfrak{i}_{\succeq}$ is evidentially proper.*

$$\mathfrak{i}_{\succeq}(p \succeq q, w) \begin{cases} a & \text{if } q \,\&\, \neg p \text{ is true in } w\text{, and } p \succ q, \\ b & \text{if } q \equiv p \text{ is true in } w\text{, and } p \succ q, \\ 1 & \text{if } p \not\equiv q \text{ is true in } w\text{, and } p \sim q, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathfrak{m}_4 = \Pr(p \,\&\, q)$, $\mathfrak{m}_3 = \Pr(\neg p \,\&\, q)$, and $\mathfrak{m}_2 = \Pr(p \,\&\, \neg q)$. Then, the following claim (which implicitly *universally quantifies* over these three probability masses $\mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4$, and presupposes that they are each on $[0, 1]$ and that they sum to at most 1) states that this parametric definition of $\mathfrak{i}_{\succeq}$ is *evidentially proper*.

$$\mathfrak{m}_2 + \mathfrak{m}_4 > \mathfrak{m}_3 + \mathfrak{m}_4 \Rightarrow \left(\begin{array}{c} a \cdot \mathfrak{m}_3 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \leq a \cdot \mathfrak{m}_2 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \\ \& \\ a \cdot \mathfrak{m}_3 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \leq \mathfrak{m}_2 + \mathfrak{m}_3 \end{array} \right)$$

$$\&$$

$$\mathfrak{m}_2 + \mathfrak{m}_4 = \mathfrak{m}_3 + \mathfrak{m}_4 \Rightarrow \left(\begin{array}{c} \mathfrak{m}_2 + \mathfrak{m}_3 \leq a \cdot \mathfrak{m}_2 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \\ \& \\ \mathfrak{m}_2 + \mathfrak{m}_3 \leq a \cdot \mathfrak{m}_3 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \end{array} \right)$$

There is a *unique* numerical assignment to the parameters a and b which makes this universal claim true, and it is $a := 2; b := 0$. This implies that our scoring rule is (essentially) the *unique* evidentially proper scoring rule for comparative confidence judgments. It also implies that there is no (truth-directed — *fn.* **??**) proper scoring rule for the weaker notion of inaccuracy associated with the assumption of a unique vindicated ordering $\mathring{\succeq}_w$ (*fn.* 8).[28] We discovered this result using *Mathematica*'s quantifier elimination algorithm for the theory of real closed fields (see the companion *Mathematica* notebook for the verification of this theorem). ◻

Verifying Theorem 7
[$(\mathcal{R}_{\succeq}) \not\Rightarrow (\text{WADA}_{\succeq})$ — for the *Kemeny measure* of inaccuracy for $\succeq$-relations]

Consider a Boolean algebra $\mathcal{B}_8$ generated by three states $\{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3\}$. And, consider the comparative confidence relation $\succeq$ on $\mathcal{B}_8$, the adjacency matrix of which is depicted in Table 5.

This relation $\succeq$ is fully represented by the regular probability function $\Pr_{\mathfrak{m}}(\cdot)$ determined by the following probability mass function:

$\mathfrak{m}_1 = 1/16$.

$\mathfrak{m}_2 = 5/16$.

$\mathfrak{m}_3 = 5/8$.

The relation $\succeq$ is weakly dominated in Kemeny distance from vindication by the relation $\succeq'$, the adjacency matrix of which is depicted in Table 6.

The relation $\succeq'$ is fully represented by the regular probability function $\Pr_{\mathfrak{m}'}(\cdot)$ determined by the following probability mass function:

[28] Similar negative results regarding the non-existence of proper scoring rules have recently been proven for imprecise probabilities (**?**).

$\succeq$	$\bot$	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_1 \vee \mathfrak{s}_2$	$\mathfrak{s}_1 \vee \mathfrak{s}_3$	$\mathfrak{s}_2 \vee \mathfrak{s}_3$	$\top$
$\bot$	1	0	0	0	0	0	0	0
$\mathfrak{s}_1$	1	1	0	0	0	0	0	0
$\mathfrak{s}_2$	1	1	1	0	0	0	0	0
$\mathfrak{s}_3$	1	1	1	1	1	0	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_2$	1	1	1	0	1	0	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_3$	1	1	1	1	1	1	0	0
$\mathfrak{s}_2 \vee \mathfrak{s}_3$	1	1	1	1	1	1	1	0
$\top$	1	1	1	1	1	1	1	1

Table 5: Adjacency matrix of a fully Pr-representable relation $\succeq$ that is weakly dominated in Kemeny distance from vindication.

$\succeq'$	$\bot$	$\mathfrak{s}_1$	$\mathfrak{s}_2$	$\mathfrak{s}_3$	$\mathfrak{s}_1 \vee \mathfrak{s}_2$	$\mathfrak{s}_1 \vee \mathfrak{s}_3$	$\mathfrak{s}_2 \vee \mathfrak{s}_3$	$\top$
$\bot$	1	0	0	0	0	0	0	0
$\mathfrak{s}_1$	1	1	1	0	0	0	0	0
$\mathfrak{s}_2$	1	1	1	0	0	0	0	0
$\mathfrak{s}_3$	1	1	1	1	0	0	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_2$	1	1	1	1	1	0	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_3$	1	1	1	1	1	1	1	0
$\mathfrak{s}_2 \vee \mathfrak{s}_3$	1	1	1	1	1	1	1	0
$\top$	1	1	1	1	1	1	1	1

Table 6: Adjacency matrix of a fully Pr-representable $\succeq'$ that weakly dominates $\succeq$ (Table 5) in Kemeny distance from vindication.

$$\mathfrak{m}'_1 = {}^{7}/_{24}.$$

$$\mathfrak{m}'_2 = {}^{7}/_{24}.$$

$$\mathfrak{m}'_3 = {}^{5}/_{12}.$$

Let w_i be the possible world corresponding to state $\mathfrak{s}_i$ of $\mathcal{B}_8$. It is then straightforward to verify that $\succeq'$ weakly dominates $\succeq$ in Kemeny inaccuracy (*viz.*, δ-inaccuracy), *i.e.*, that the following claims are true (see the companion *Mathematica* notebook).

(*i*) $(\forall w_i)\,[\delta(\succeq', w_i) \leq \delta(\succeq, w_i)]$, and

(*ii*) $(\exists w_i)\,[\delta(\succeq', w_i) < \delta(\succeq, w_i)]$.

Therefore, because probabilistic representability of $\succeq$ (even by a *regular* probability function) is not sufficient to ensure that $\succeq$ is not weakly dominated in Kemeny inaccuracy, the Kemeny measure of inaccuracy δ is not evidentially proper (even for regular evidential probability functions). This can be seen by calculating the expected Kemeny inaccuracy of $\succeq$ and $\succeq'$ above, relative to the evidential representer of $\succeq$ ($\Pr_{\mathfrak{m}}$). In the companion *Mathematica* notebook, we perform these calculations, which reveal that $\succeq'$ has a lower expected Kemeny inaccuracy than $\succeq$, relative to $\Pr_{\mathfrak{m}}$. Of course, this is not surprising, since it would be true for *any* regular probability function. $\square$