Toward an Epistemic Foundation for Comparative Confidence

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Aim: give epistemic justifications of coherence requirements for ≥ that have appeared in the contemporary literature.

Means: exploit a generalization of Joyce’s non-pragmatic argument for probabilism [18, 19]. Note: something similar has already been done for full belief [10, 1, 8, 13].

Joyce was inspired by an elegant geometrical argument of de Finetti [5] (see Extras). However, unlike de Finetti, Savage, et. al. [24, 15, 17] Joyce’s approach is epistemic in nature.

Abstracting away from Joyce’s argument, we have developed a framework [13] for grounding epistemic coherence requirements for judgment sets J = {j1, . . . , jn} (of type J) over agendas of propositions A ={p1, . . . , pn}.

Applying our framework involves three steps.

Step 1: Identify a precise sense in which individual judgments j of type J can be (qualitatively) inaccurate (or alethically defective/imperfect) at a possible world w.
We begin with some background assumptions about ≥.

Our first assumption is that our agents S form comparative confidence judgments ≥ regarding all pairs of propositions on some m-proposition agenda A, drawn from some n-proposition Boolean algebra B_n (m ≤ n, viz., A ≤ B_n).

Our second assumption is that ≥ is a total preorder on A, i.e., ≥ satisfies the following conditions, for all p, q, r ∈ A.

**Totality.** (p ≥ q) ∨ (q ≥ p).

**Transitivity.** If p ≥ q and q ≥ r, then p ≥ r.

Global versions of these are controversial [14, 12, 23]. We’re only assuming local versions of them (for some agendas A).

Once we’ve got a total preorder ≥ on A, we can then define a “strictly more confident than” relation on A, as follows.

p > q ⇔ p ≥ q and q ∗ p.

Because ≥ is a total preorder on A, it will follow that > is an asymmetric, transitive, irreflexive relation on A.

We can also define an “equally confident in” (or “epistemically indifferent between”) relation on A, as:

p ∼ q ⇔ p ≥ q and q ≥ p.

Since ≥ is a total preorder, ∼ is an equivalence relation.

Next, we’ll assume our agents S are logically omniscient.

(LO) S respects all logical equivalencies.

: If p, q are logically equivalent, then S judges p ∼ q. And, if S judges p > q, then p, q are not logically equivalent.

Finally, we’ll assume our agents S have regular ≥-orderings.

**Regularity.** If p is contingent, then p > ⊥ and ⊥ > p.

We can represent ≥-relations on agendas A via their 0/1 adjacency matrices A^≥, where A^≥_{ij} = 1 iff p_i ≥ p_j.

Toy example: let A = B_4 be the smallest sentential BA, with four propositions {⊥, P, ¬P, ⊤}, for some contingent P. Specifically, interpret P as “a tossed coin lands heads.”

The above figure shows the adjacency matrix and graphical representation of a relation (≥') on B_4. This relation ≥' is supported by S’s evidence E, if E says that the coin is fair.

Consider an alternative relation (≥’) on B_4, which agrees with ≥ on all judgments, except for ¬P ≥ P. That is, P >’ ¬P; whereas, P ∼ ¬P. [≥’ is depicted on the next slide.]

This alternative relation ≥’ on B_4 is supported by S’s evidence E, if E says that the coin is biased toward heads.

Intuitively, neither ≥ nor ≥’ should be deemed (formally) incoherent. After all, either could be supported by an agent’s evidence. We’ll return to evidential requirements for comparative confidence relations below. Meanwhile, Step 1.
\textbf{Step 1} involves articulating a precise sense in which an individual comparative confidence judgment \( p \geq q \) is inaccurate at \( w \). Here, we follow Joyce’s [18, 19] extensionality assumption, which requires “inaccuracy” to supervene on the truth-values of the propositions in \( \mathcal{A} \) at \( w \).

An individual comparative confidence judgment \( p \geq q \) is inaccurate at \( w \) iff \( p \geq q \) entails that the ordering \( \geq \) fails to rank all truths strictly above all falsehoods at \( w \).

On this conception, there are two facts about the inaccuracy of individual comparative confidence judgments \( p \geq q \).

\textbf{Fact 1.} If \( q \& \neg p \) is true at \( w \), then \( p > q \) is inaccurate at \( w \).

\textbf{Fact 2.} If \( p \neq q \) is true at \( w \), then \( p \sim q \) is inaccurate at \( w \).

\footnote{One might be tempted by a weaker (and “more Joycean”) definition of inaccuracy, according to which \( p \geq q \) is inaccurate if it contradicts \( p \sim q \) induced by the indicator function \( v_w \). This weaker definition (which also deems \( p \geq q \) inaccurate if \( p \equiv q \) is true at \( w \)) is untenable for us. This will follow from our Fundamental Theorem, below.}

\textbf{Step 2} requires a point-wise inaccuracy measure \( i(p \geq q, w) \).

There are two kinds of inaccurate \( \geq \)-judgments (Facts 1 and 2). Intuitively, these two kinds of inaccuracies should not receive equal \( i \)-scores. Mistaken \( > \)-judgments should receive greater \( i \)-scores than mistaken \( \sim \)-judgments.

\textbf{How much more inaccurate} than \( \sim \) mistakes are \( > \) mistakes? Twice as inaccurate! Suppose (by convention) that we assign an \( i \)-score of 1 to mistaken \( \sim \)-judgments. We must (!) assign an \( i \)-score of 2 to mistaken \( > \)-judgments.

\[ i(p \geq q, w) = \begin{cases} 2 & \text{if } q \& \neg p \text{ is true at } w, \text{ and } p > q, \\ 1 & \text{if } p \neq q \text{ is true at } w, \text{ and } p \sim q, \\ 0 & \text{otherwise}. \end{cases} \]

\( \geq \)'s total inaccuracy (on \( \mathcal{A} \) at \( w \)) is the sum of \( \geq \)'s \( i \)-scores.

\[ I(\geq, w) = \sum_{p,q \in \mathcal{A}} i(p \geq q, w). \]
• Two kinds of representability of $\geq$, by a real-valued $f$.
  • $\geq$ is fully represented by $f \equiv$ for all $p, q \in B_n$
    $p \geq q \iff f(p) \geq f(q)$.
  • $\geq$ is partially represented by $f \equiv$ for all $p, q \in B_n$
    $p > q \rightarrow f(p) > f(q)$.

- Now, (C) can be expressed equivalently, as follows:
  (C) $S$’s $\geq$-relation (assumed to be a total preorder on $B_n$)
       should be fully representable by some plausibility measure.

\[ \textbf{Theorem 1.} \text{(WADA) entails (C). [See Extras for a proof.]} \]

- There are several other coherence requirements for $\geq$ that
  can be expressed both axiomatically, and in terms of
  numerical representability by some real-valued $f$.

- We’ll state these, and say whether or not they follow from
  (WADA). The next requirements involve belief functions.

\[ \textbf{Theorem 2.} \text{(A$_1$) $(\geq)$ must be a total preorder on $B_n$} \]

\[ \textbf{Theorem 3.} \text{(WADA) entails (C$_1$). [See Extras.]} \]

- Moving beyond (C$_1$) takes us into comparative probability. A
t.p. $\geq$ is a comparative probability iff $\geq$ satisfies (A$_1$), (A$_2$), &
  (A$_5$) If $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive, then:
    $q > p \rightarrow q \lor r > p \lor r$

\[ \textbf{Theorem 4.} \text{(WADA) does not entail (C$_2$). [See Extras.]} \]

- The following axiomatic constraint is a weakening of (A$_5$).
  (A$_5'$) If $\langle p, q \rangle$ and $\langle p, r \rangle$ are mutually exclusive, then:
    $q > r \Rightarrow p \lor r \equiv p \lor q$

- And, the following coherence requirement is a
  (corresponding) weakening of coherence requirement (C$_2$).
  (C$_2'$) $\geq$ should be a total preorder and satisfy (A$_1$), (A$_2$) and (A$_5'$).

\[ \textbf{Theorem 5.} \text{(WADA) does not entail (C$_2'$). [See Extras.]} \]

- Our final pair of coherence requirements for $\geq$ involve
  representability by some probability function.

- I’m sure everyone knows what a Pr-function is, but...

- Probability functions are special kinds of belief functions
  (just as belief functions were special kinds of PI-measures).
A probability mass function is a function $m$ which maps states of $B_n$ to $[0, 1]$, and which satisfies these two axioms.

\begin{itemize}
  \item (B1) $m(\bot) = 0$.
  \item (B2) $\sum_{s \in B_n} m(s) = 1$.
\end{itemize}

A probability function $Pr : B_n \rightarrow [0, 1]$ is generated by an underlying probability mass function $m$ in the following way

$$Pr_m(p) \equiv \sum_{s \in B_n} m(s).$$

That brings us to our final pair of requirements for $\succeq$.

\begin{itemize}
  \item (C3) $\succeq$ should be be be partially representable by some Pr-function.
  \item (C4) $\succeq$ should be fully representable by some Pr-function.
\end{itemize}

de Finetti [3, 4] famously conjectured that (C2) entails (C4). But, Kraft et. al. [22] showed that (C2) $\nRightarrow$ (C3). [See Extras.]

We have the following logical relations between the $C$'s.

- full rep. by Pr
  $$(C_0) \xrightarrow{\neg} (C_1)$$
- partial rep. by Pr
  $$(C_2) \xrightarrow{\neg} (C_4)$$
- qualitative prob.
  $$(C_2) \cong (C_3)$$
- (A$_1$) + (A$_2$) + (A$^*_5$)

If a requirement follows from (WADA), it gets a “$\triangleright$”. If a requirement does not follow from (WADA), it gets an “$\nRightarrow$.”

We conclude with our final (and most important) Fundamental Theorem(s). [See Extras for proofs.]

**Theorem 1.** (WADA) entails (C), viz., (WADA) $\Rightarrow$ (A$_1$) & (A$_2$).

**Proof.**

Suppose $\succeq$ violates (A$_1$). Because $\succeq$ is total, this means $\succeq$ is such that $\bot \succeq \top$. Consider the relation $\succeq'$ which agrees with $\succeq$ on all comparisons outside the $(\bot, \top)$-fragment, but which is such that $\top \succeq' \bot$. We have: $$(\forall w) \{i(\top \succeq' \bot, w) = 0 < 1 \leq i(\bot \succeq \top, w)\}. \quad \square$$

Suppose $\succeq$ violates (A$_2$). Because $\succeq$ is total, this means there is a pair of propositions $p$ and $q$ in $A$ such that (a) $p$ entails $q$ but (b) $p \nRightarrow q$. Consider the relation $\succeq'$ which agrees with $\succeq$ outside of the $(p, q)$-fragment, but which is such that $q \succeq' p$. The table on the next slide depicts the $(p, q)$-fragments of the relations $\succeq$ and $\succeq'$ in the three salient possible worlds $w_1$-$w_3$ not ruled out by (a) $p \nRightarrow q$. By (b) & (LO), $p$ and $q$ are not logically equivalent. So, world $w_2$ is a live possibility, and $\succeq'$ weakly I-dominates $\succeq$. \quad \square
### General Background

There are only four alternative judgment sets that need to be shown (by exhaustive search) that $I(\succeq, w_i) \wedge I(\succeq', w_i)$.

<table>
<thead>
<tr>
<th>$w_i$</th>
<th>$p$</th>
<th>$q$</th>
<th>$\succeq$</th>
<th>$\succeq'$</th>
<th>$I(\succeq, w_i)$</th>
<th>$I(\succeq', w_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>T</td>
<td>T</td>
<td>$p \succ q$</td>
<td>$q \succ' p$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$w_2$</td>
<td>T</td>
<td>F</td>
<td>$p \succ q$</td>
<td>$q \succ' p$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$w_3$</td>
<td>F</td>
<td>F</td>
<td>$p \succ q$</td>
<td>$q \succ' p$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Proof.

Having already proved Theorem 1, we just need to show that (WADA) entails (A3). Suppose $\succeq$ violates (A3). Because $\succeq$ is total, this means there must exist $p, q, r \in A$ such that (a) $p \equiv q$, (b) $(q, r)$ are mutually exclusive, (c) $q > p$, but (d) $p \lor r \equiv q \lor r$. Let $\succeq'$ agree with $\succeq$ on every judgment, except (d). That is, let $\succeq'$ be such that (e) $q \succ' p$ and (f) $q \lor r \succ' p \lor r$. There are only four worlds $(p, q, r)$ state descriptions) compatible with the precondition of (A3). These are the following (state descriptions).

- $w_1 = p \land q \land \neg r$
- $w_2 = \neg p \land q \land \neg r$
- $w_3 = \neg p \land \neg q \land r$
- $w_4 = \neg p \land \neg q \land \neg r$

By (c) & (LO), $p$ and $q$ are not logically equivalent. As a result, world $w_2$ is a live possibility. Moreover, (f) will not be inaccurate in any of these four worlds. But, (d) must be inaccurate in world $w_2$. This suffices to show that $\succeq'$ weakly $I$-dominates $\succeq$. $\Box$

### Theorem 3.

(WADA) entails $(C_1)$.

### Proof.

Suppose $\Pr(\cdot)$ fully represents $\succeq$. Consider the expected $I$-inaccuracy, as calculated by $\Pr(\cdot)$, of $\succeq$: $\mathbb{E}_{\Pr}^I = \sum_{w \in A} \Pr(w) \cdot I(\succeq, w)$. Since $I(\succeq, w)$ is a sum of the $\i(p \succeq q, w)$ for each $(p, q) \in A$, and since $\mathbb{E}$ is linear:

$$\mathbb{E}_{\Pr}^I = \sum_{p, q \in A} \Pr(p \succeq q, w)$$

(1) Suppose $\Pr(p) > \Pr(q)$. Then we have:

$$\Pr(p \succeq q, w) = 2 \cdot \Pr(q \land \neg p) < \Pr(p \succeq q, w) = \Pr(q \land \neg p), \text{ and}$$

$$\Pr(p \succeq q, w) = 2 \cdot \Pr(p \land \neg q) < \Pr(p \succeq q, w) = 2 \cdot \Pr(p \land \neg q).$$

(2) Suppose $\Pr(p) = \Pr(q)$. Then we have:

$$\Pr(p \succeq q, w) = \Pr(p \succeq q) = \Pr(p \succeq q, w) = 2 \cdot \Pr(q \land \neg p).$$

As a result, if $\succeq$ is fully representable by any $\Pr(\cdot)$, then $\succeq$ cannot be strictly $I$-dominated, i.e., $(C_1) \Rightarrow (SADA)$. Moreover, if we assume $\Pr(\cdot)$ to be regular, then $\succeq$ must satisfy (WADA) [13]. $\Box$: $(R) \Rightarrow (WADA)$. $\Box$
**Theorem.** $a := 2; b := 0$ is the only assignment to $a, b$ that ensures the following definition of $i$ is evidentially proper.

$$i(p \geq q, w) \equiv \begin{cases} a & \text{if } q \land \neg p \text{ is true in } w, \text{ and } p > q, \\ b & \text{if } q \equiv p \text{ is true in } w, \text{ and } p > q, \\ 1 & \text{if } p \neq q \text{ is true in } w, \text{ and } p \sim q, \\ 0 & \text{otherwise.} \end{cases}$$

Let $m_4 = \Pr(p \land q), m_3 = \Pr(\neg p \land q)$, and $m_2 = \Pr(p \land \neg q)$. Then, the propriety of $i$ is equivalent to the following (universal) claim. And, the only assignment that makes this (universal) claim true is $a := 2; b := 0$.  

$$m_2 + m_4 > m_3 + m_4 \Rightarrow \left\{ \begin{array}{l} a \cdot m_3 + b \cdot (1 - (m_2 + m_3)) \leq a \cdot m_2 + b \cdot (1 - (m_2 + m_3)) \\ a \cdot m_3 + b \cdot (1 - (m_2 + m_3)) \leq m_2 + m_3 \end{array} \right.$$

$$m_2 + m_4 = m_3 + m_4 \Rightarrow \left\{ \begin{array}{l} m_2 + m_3 \leq a \cdot m_2 + b \cdot (1 - (m_2 + m_3)) \\ m_2 + m_3 \leq a \cdot m_3 + b \cdot (1 - (m_2 + m_3)) \end{array} \right.$$

In their seminal paper, Kraft et al. [22] refute de Finetti’s [3, 4] conjecture: $(\mathcal{C}_2) \Rightarrow (\mathcal{C}_4)$. In fact, they show $(\mathcal{C}_2) \nRightarrow (\mathcal{C}_4)$.

Their counterexample involves a linear order $\succeq$ on an algebra $\mathcal{B}_{3,2}$ generated by five states: $\{s_1, \ldots, s_5\}$.

We won’t write down the entire linear order $\succeq$ as this involves a complete ranking of 32 propositions. Instead, we focus only the following, salient 8-proposition fragment.

<table>
<thead>
<tr>
<th>$\succeq$</th>
<th>$s_1$</th>
<th>$s_2 \lor s_4$</th>
<th>$s_1 \lor s_2$</th>
<th>$s_2 \lor s_5$</th>
<th>$s_1 \lor s_2 \lor s_3$</th>
<th>$s_1 \lor s_2 \lor s_4$</th>
<th>$s_1 \lor s_2 \lor s_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_2 \lor s_4$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_1 \lor s_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_1 \lor s_2 \lor s_3$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_1 \lor s_2 \lor s_4$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>$s_1 \lor s_2 \lor s_5$</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Simplest case of dF’s Theorem [5]:** $b(P) = x; b(\neg P) = y$. The diagonal lines are the probabilistic $b$’s (on $(P, \neg P)$).

The two directions of de Finetti’s theorem (for $(P, \neg P)$) can be established via these two figures. And, this simplest $(P, \neg P)$ version of the Theorem generalizes from the simplest propositional Boolean algebra $\mathcal{B}_4$ to $\mathcal{B}_n$, for any $n$. 
There are two, weaker \( \succsim \)-dominance requirements that we discuss in the book [13]. These are as follows.

**Strict Accuracy-Dominance Avoidance (SADA).** \( \succsim \) should *not be strictly dominated* in inaccuracy (according to \( T \)). More formally, there should not exist a \( \succsim' \) (on \( \mathcal{A} \)) such that

\[
(\forall w) \ [T(\succsim', w) < T(\succsim, w)].
\]

- Of course, (SADA) is *strictly weaker* than (WADA). And, here is a requirement that is *even weaker* than (SADA).

- Let \( M(\succsim, w) \equiv \text{the set of } \succsim \text{'s inaccurate judgments at } w \).

**Strong Strict Accuracy-Dominance Avoidance (SSADA).** There should not exist a \( \succsim' \) on \( \mathcal{A} \) such that:

\[
(\forall w) \ [M(\succsim', w) \subset M(\succsim, w)].
\]

- Some of our (WADA) results also *go through* for (SADA) and/or (SSADA). Finally, we give a complete, "big picture" of all the logical relations among all the requirements.

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**References**


