Toward an Epistemic Foundation for Comparative Confidence

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- The contemporary literature focuses mainly on two types of *non-comparative* judgment: belief and credence. Not much attention is paid to *comparative* judgment (but see [16]).
- It wasn't always thus. Keynes [21], de Finetti [3, 4] and Savage [24] all emphasized the importance (and perhaps even *fundamentality*) of comparative confidence.
- *Comparative confidence* is a three-place relation between an agent S (at a time t) and a pair of propositions $\langle p, q\rangle$.
- We'll use ⌜$p \succeq q$⌝ to express this relation, *viz.*, ⌜S is at least as confident in the truth of p as she is in the truth of q⌝.
- It is difficult to articulate the meaning of $\succeq$ without somehow implicating that it essentially involves some *non-comparative* judgments [*e.g.*, $b(p) \geq b(q)$].

☞ But, it's important to think of $\succeq$ as *autonomous* and *irreducibly comparative — i.e.*, as a kind of comparative judgment *that may not reduce to anything non-comparative.*

- **Aim**: give *epistemic justifications* of coherence requirements for $\succeq$ that have appeared in the contemporary literature.
- **Means**: exploit a generalization of Joyce's non-pragmatic argument for probabilism [18, 19]. Note: something similar has already been done for full belief [10, 1, 8, 13].
- Joyce was inspired by an elegant geometrical argument of de Finetti [5] (see Extras). However, unlike de Finetti, Savage, *et. al.* [24, 15, 17] Joyce's approach is *epistemic* in nature.
- Abstracting away from Joyce's argument, we have developed a *framework* [13] for grounding epistemic coherence requirements for judgment sets $\mathbf{J} = \{j_1, \ldots, j_n\}$ (of type $\mathfrak{J}$) over *agendas* of propositions $\mathcal{A} = \{p_1, \ldots, p_n\}$.
- Applying our framework involves **three steps**.
  - **Step 1**: Identify a precise sense in which individual judgments j of type $\mathfrak{J}$ can be (qualitatively) *inaccurate* (or *alethically defective/imperfect*) at a possible world w.

  - **Step 2**: Define an *inaccuracy score* $\mathfrak{i}(j, w)$ for individual judgments j of type $\mathfrak{J}$. This is a numerical measure of *how inaccurate* (in the sense of Step 1) j is (at w). For each *set* $\mathbf{J} = \{j_1, \ldots, j_n\}$, we define its *total inaccuracy* at w as the *sum* of the $\mathfrak{i}$-scores of its members: $\mathcal{I}(\mathbf{J}, w) \overset{\text{def}}{=} \sum_i \mathfrak{i}(j_i, w)$.
  - **Step 3**: Adopt a *fundamental epistemic principle*, which uses $\mathcal{I}(\mathbf{J}, w)$ to ground a (formal, synchronic, epistemic) coherence requirement for judgment sets $\mathbf{J}$ of type $\mathfrak{J}$.
- In the case of Joyce's argument for probabilism, we have:

  Step 1: ⌜$b(p) = r$⌝ is *inaccurate* at w just in case r differs from the value assigned to p by the *indicator function* $v_w(p)$, which is 1 (0) if p is true (false) at w.

  Step 2: $\mathfrak{i}(b(p), w)$ is (squared) *Euclidean distance* (or Brier score) between $b(p)$ and $v_w(p)$. $\mathcal{I}(b, w) = \sum_i \mathfrak{i}(b(p_i), w)$.

  Step 3: The *fundamental epistemic principle*: b shouldn't be *weakly dominated* (by any b'), according to $\mathcal{I}(\cdot, w)$.
- Today: we apply the framework to *comparative confidence*.

- We begin with some background assumptions about $\succeq$.
- Our first assumption is that our agents S form comparative confidence judgments $\succeq$ regarding all pairs of propositions on some m-proposition *agenda* $\mathcal{A}$, drawn from some n-proposition Boolean algebra $\mathcal{B}_n$ ($m \leq n$, *viz.*, $\mathcal{A} \subseteq \mathcal{B}_n$).
- Our second assumption is that $\succeq$ is a *total preorder* on $\mathcal{A}$, *i.e.*, $\succeq$ satisfies the following conditions, for all $p, q, r \in \mathcal{A}$.

  Totality. $(p \succeq q) \vee (q \succeq p)$.

  Transitivity. If $p \succeq q$ and $q \succeq r$, then $p \succeq r$.
- *Global* versions of these are controversial [14, 12, 23]. We're only assuming *local* versions of them (for *some* agendas $\mathcal{A}$).
- Once we've got a total preorder $\succeq$ on $\mathcal{A}$, we can then define a "strictly more confident than" relation on $\mathcal{A}$, as follows.

  $p \succ q \overset{\text{def}}{=} p \succeq q$ and $q \not\succeq p$.
- Because $\succeq$ is a total preorder on $\mathcal{A}$, it will follow that $\succ$ is an *asymmetric, transitive, irreflexive* relation on $\mathcal{A}$.

- We can also define an "equally confident in" (or "epistemically indifferent between") relation on $\mathcal{A}$, as:

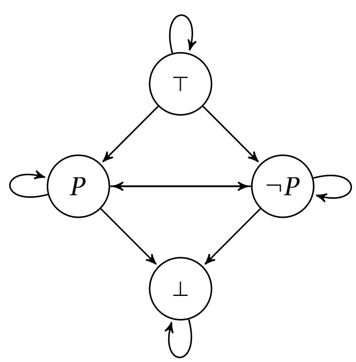
  $p \sim q \overset{\text{def}}{=} p \succeq q$ and $q \succeq p$.
- Since $\succeq$ is a total preorder, $\sim$ is an *equivalence relation.*
- Next, we'll assume our agents S are *logically omniscient.*

  (LO) S respects all logical equivalencies.

  ☞ $\therefore$ If p, q are logically equivalent, then S judges $p \sim q$. And, if S judges $p \succ q$, then p, q are *not* logically equivalent.
- Finally, we'll assume our agents S have *regular* $\succeq$-orderings.

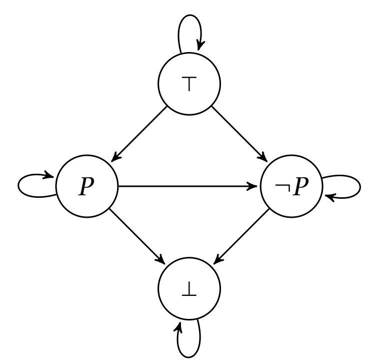
  Regularity. If p is contingent, then $p \succ \bot$ and $\top \succ p$.
- We can represent $\succeq$-relations on agendas $\mathcal{A}$ *via* their 0/1 *adjacency matrices* $A^{\succeq}$, where $A^{\succeq}_{ij} = 1$ *iff* $p_i \succeq p_j$.
- Toy example: let $\mathcal{A} = \mathcal{B}_4$ be the smallest sentential BA, with four propositions $\langle \top, P, \neg P, \bot \rangle$, for some contingent P. Specifically, interpret P as "a tossed coin lands heads."

$\succeq$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	1	1	1
$\bot$	0	0	0	1



- The above figure shows the adjacency matrix and graphical representation of a relation ($\succeq$) on $\mathcal{B}_4$. This relation $\succeq$ is *supported by S's evidence E*, ***if*** E says that the coin is *fair.*
- Consider an alternative relation ($\succeq'$) on $\mathcal{B}_4$, which agrees with $\succeq$ on all judgments, *except for* $\neg P \succeq P$. That is, $P \succ' \neg P$; whereas, $P \sim \neg P$. [$\succeq'$ is depicted on the next slide.]

$\succeq'$	$\top$	P	$\neg P$	$\bot$
$\top$	1	1	1	1
P	0	1	1	1
$\neg P$	0	0	1	1
$\bot$	0	0	0	1



- This alternative relation $\succeq'$ on $\mathcal{B}_4$ is *supported by S's evidence E*, ***if*** E says that the coin is *biased toward heads.*
- Intuitively, neither $\succeq$ nor $\succeq'$ should be deemed (formally) *incoherent.* After all, either *could be supported by an agent's evidence.* We'll return to evidential requirements for comparative confidence relations below. Meanwhile, Step 1.

- **Step 1** involves articulating a precise sense in which an individual comparative confidence judgment $p \succeq q$ is *inaccurate* at w. Here, we follow Joyce's [18, 19] *extensionality* assumption, which requires "inaccuracy" to *supervene on the truth-values of the propositions in $\mathcal{A}$ at w*.

☞ An individual comparative confidence judgment $p \succeq q$ is *inaccurate at w* iff *$p \succeq q$ entails that the ordering $\succeq$ fails to rank all truths strictly above all falsehoods at w*.[1]

- On this conception, there are *two facts* about the inaccuracy of individual comparative confidence judgments $p \succeq q$.

  Fact 1. If $q \mathbin{\&} \neg p$ is true at w, then $p \succ q$ is inaccurate at w.

  Fact 2. If $p \not\equiv q$ is true at w, then $p \sim q$ is inaccurate at w.

[1]One might be tempted by a weaker (and "more Joycean") definition of inaccuracy, according to which $p \succeq q$ is inaccurate iff it *contradicts* the comparison $p \succeq_w q$ *induced by the indicator function v_w*. This weaker definition (which *also* deems $p \succ q$ inaccurate *if $p \equiv q$ is true at w*) is *untenable* for us. This will follow from our Fundamental Theorem, below.

- **Step 2** requires a *point-wise* inaccuracy measure $\mathfrak{i}(p \succeq q, w)$.

☞ There are two kinds of inaccurate $\succeq$-judgments (Facts 1 and 2). *Intuitively, these two should kinds of inaccuracies should not receive equal $\mathfrak{i}$-scores.* Mistaken $\succ$ judgments should receive *greater $\mathfrak{i}$-scores* than mistaken $\sim$ judgments.

- *How much more inaccurate* than $\sim$ mistakes are $\succ$ mistakes? *Twice as inaccurate!* Suppose (by convention) that we assign an $\mathfrak{i}$-score of 1 to mistaken $\sim$ judgments. We *must* (!) assign an $\mathfrak{i}$-score of 2 to mistaken $\succ$ judgments.

$$\mathfrak{i}(p \succeq q, w) \stackrel{\text{def}}{=} \begin{cases} 2 & \text{if } q \mathbin{\&} \neg p \text{ is true at } w \text{, and } p \succ q, \\ 1 & \text{if } p \not\equiv q \text{ is true at } w \text{, and } p \sim q, \\ 0 & \text{otherwise.} \end{cases}$$

- $\succeq$'s *total inaccuracy* (on $\mathcal{A}$ at w) is the *sum* of $\succeq$'s $\mathfrak{i}$-scores.

$$\mathcal{I}(\succeq, w) \stackrel{\text{def}}{=} \sum_{p,q \in \mathcal{A}} \mathfrak{i}(p \succeq q, w).$$

- **Step 3** involves the adoption of a *fundamental epistemic principle*. Here, we will follow Joyce and adopt:

  Weak Accuracy-Dominance Avoidance (WADA). $\succeq$ should *not be weakly dominated* in inaccuracy (according to $\mathcal{I}$). More formally, there should *not* exist a $\succeq'$ (on $\mathcal{A}$) such that

  (i) $(\forall w)\,[\mathcal{I}(\succeq', w) \leq \mathcal{I}(\succeq, w)]$, and

  (ii) $(\exists w)\,[\mathcal{I}(\succeq', w) < \mathcal{I}(\succeq, w)]$.

- Recall our toy relations $\succeq$ and $\succeq'$ over $\mathcal{B}_4$. Neither of these relations should be *ruled-out as incoherent*, as each *could be* supported by *some* body of evidence [19, *pp.* 282–3].

☞ **Theorem**. *Neither $\succeq$ nor $\succeq'$ is weakly dominated in $\mathcal{I}$-inaccuracy — by **any** binary relation on $\mathcal{B}_4$.*

- This result is a corollary of our Fundamental Theorem, which will also explain why we were *forced* to assign an inaccuracy score of *exactly 2* to inaccurate $\succ$ judgments.

- More on that later. Meanwhile, a historical interlude.

- Various coherence requirements for $\succeq$ have been discussed [15, 2, 26]. We'll focus on a *particular family* of these.

- We begin with *the fundamental requirement* ($\mathfrak{C}$), which has (near) universal acceptance. We will state ($\mathfrak{C}$) in two ways: axiomatically, and in terms of numerical representability.

  ($\mathfrak{C}$) S's $\succeq$-relation (assumed to be a total preorder on $\mathcal{B}_n$) should satisfy the following two axiomatic constraints:

  (A_1) $\top \succ \bot$.

  (A_2) For all $p, q \in \mathcal{B}_n$, if p entails q then $q \succeq p$.

- A *plausibility measure* (*a.k.a.*, a *capacity*) on a Boolean algebra $\mathcal{B}_n$ is real-valued function $\mathrm{Pl} : \mathcal{B}_n \mapsto [0, 1]$ which satisfies the following three conditions [15, *p.* 51]:

(Pl_1) $\mathrm{Pl}(\bot) = 0$.

(Pl_2) $\mathrm{Pl}(\top) = 1$.

(Pl_3) For all $p, q \in \mathcal{B}_n$, if p entails q then $\mathrm{Pl}(q) \geq \mathrm{Pl}(p)$.

- Two kinds of representability of $\succeq$, by a real-valued f.
  - $\succeq$ is *fully* represented by $f \overset{\text{def}}{=}$ for all $p, q \in \mathcal{B}_n$

$$p \succeq q \iff f(p) \geq f(q).$$

  - $\succeq$ is *partially* represented by $f \overset{\text{def}}{=}$ for all $p, q \in \mathcal{B}_n$

$$p \succ q \implies f(p) > f(q).$$

- Now, ($\mathfrak{C}$) can be expressed equivalently, as follows:

($\mathfrak{C}$) S's $\succeq$-relation (assumed to be a total preorder on $\mathcal{B}_n$) should be *fully representable by some plausibility measure.*

☞ **Theorem 1**. (WADA) entails ($\mathfrak{C}$). [See Extras for a proof.]

- There are several other coherence requirements for $\succeq$ that can be expressed both axiomatically, and in terms of numerical representability by some real-valued f.
- We'll state these, and say whether or not they follow from (WADA). The next requirements involve *belief functions.*

- A *mass function* on a Boolean algebra $\mathcal{B}_n$ is a function $m : \mathcal{B}_n \mapsto [0,1]$ that satisfies the following two conditions:

(M_1) $m(\bot) = 0$.

(M_2) $\sum_{p \in \mathcal{B}_n} m(p) = 1$.

- A *belief function* $\text{Bel} : \mathcal{B}_n \mapsto [0,1]$ is generated by an underlying mass function m on $\mathcal{B}_n$ in the following way:

$$\text{Bel}_m(p) \overset{\text{def}}{=} \sum_{\substack{q \in \mathcal{B}_n \\ q \text{ entails } p}} m(q).$$

- Now, consider the following coherence requirement:

($\mathfrak{C}_0$) S's $\succeq$-relation (assumed to be a total preorder on $\mathcal{B}_n$) should be *partially* representable by some belief function.

- A total preorder $\succeq$ satisfies ($\mathfrak{C}_0$) iff $\succeq$ satisfies (A_2) [26]. So, Theorem 1 has a Corollary: ["Thm 2"] (WADA) entails ($\mathfrak{C}_0$). What about *full* representability of a belief function? To wit:

($\mathfrak{C}_1$) S's $\succeq$-relation (assumed to be a total preorder on $\mathcal{B}_n$) should be *fully* representable by a belief function.

- As it turns out [26], a relation $\succeq$ is *fully* representable by some belief function if and only if $\succeq$ satisfies (A_1), (A_2), *and*

(A_3) If p entails q and $\langle q, r\rangle$ are mutually exclusive, then:

$$q \succ p \implies q \vee r \succ p \vee r.$$

- (WADA) also entails (A_3). That is, we have the following:

Theorem 3. (WADA) entails ($\mathfrak{C}_1$). [See Extras.]

- Moving beyond ($\mathfrak{C}_1$) takes us into *comparative probability*. A t.p. $\succeq$ is a *comparative probability* iff $\succeq$ satisfies (A_1), (A_2), &

(A_5) If $\langle p, q\rangle$ and $\langle p, r\rangle$ are mutually exclusive, then:

$$q \succeq r \iff p \vee q \succeq p \vee r$$

($\mathfrak{C}_2$) S's $\succeq$-relation (assumed to be a total preorder on $\mathcal{B}_n$) should be a *comparative probability* relation.

Theorem 4. (WADA) does *not* entail ($\mathfrak{C}_2$). [See Extras.]

- The following axiomatic constraint is a weakening of (A_5).

($A_5^\star$) If $\langle p, q\rangle$ and $\langle p, r\rangle$ are mutually exclusive, then:

$$q \succ r \implies p \vee r \nsucc p \vee q$$

- And, the following coherence requirement is a (corresponding) weakening of coherence requirement ($\mathfrak{C}_2$).

($\mathfrak{C}_2^\star$) $\succeq$ should (be a total preorder and) satisfy (A_1), (A_2) and ($A_5^\star$).

Theorem 5. (WADA) does *not* entail ($\mathfrak{C}_2^\star$). [See Extras.]

- Our final pair of coherence requirements for $\succeq$ involve representability by some *probability* function.
- I'm sure everyone knows what a Pr-function is, but…
- Probability functions are special kinds of belief functions (just as belief functions were special kinds of Pl-measures).

- A *probability* mass function is a function $\mathfrak{m}$ which maps *states* of $\mathcal{B}_n$ to $[0,1]$, and which satisfies these two axioms.

($\mathfrak{M}_1$) $\mathfrak{m}(\bot) = 0$.

($\mathfrak{M}_2$) $\sum\limits_{\mathfrak{s} \in \mathcal{B}_n} \mathfrak{m}(\mathfrak{s}) = 1$.

- A *probability function* $\Pr : \mathcal{B}_n \mapsto [0,1]$ is generated by an underlying probability mass function $\mathfrak{m}$ in the following way

$$\Pr_{\mathfrak{m}}(p) \overset{\text{def}}{=} \sum_{\substack{\mathfrak{s} \in \mathcal{B}_n \\ \mathfrak{s} \text{ entails } p}} \mathfrak{m}(\mathfrak{s}).$$

- That brings us to our final pair of requirements for $\succeq$.

($\mathfrak{C}_3$) $\succeq$ should be be *partially* representable by some Pr-function.

($\mathfrak{C}_4$) $\succeq$ should be be *fully* representable by some Pr-function.

- de Finetti [3, 4] famously conjectured that ($\mathfrak{C}_2$) entails ($\mathfrak{C}_4$). But, Kraft *et. al.* [22] showed that ($\mathfrak{C}_2$) $\not\Rightarrow$ ($\mathfrak{C}_3$). [See Extras.]

- We have the following logical relations between the $\mathfrak{C}$'s.

$$\begin{tikzcd}
\text{full rep. by Pr} & (\mathfrak{C}_4)^{\text{✗}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\mathfrak{C}_3)^{\text{✗}} \arrow[d, Rightarrow] & \text{partial rep. by Pr} \\
\text{qualitative prob.} & (\mathfrak{C}_2)^{\text{✗}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\mathfrak{C}_2^{\star})^{\text{✗}} \arrow[ddl, Rightarrow] & (\mathrm{A}_1) + (\mathrm{A}_2) + (\mathrm{A}_5^{\star}) \\
\text{full rep. by Bel} & (\mathfrak{C}_1)^{\checkmark} \arrow[d, Rightarrow] & & \\
\text{full rep. by Pl} & (\mathfrak{C})^{\checkmark} \arrow[d, Rightarrow] & & \\
\text{partial rep. by Bel} & (\mathfrak{C}_0)^{\checkmark} & &
\end{tikzcd}$$

- If a requirement follows from (WADA), it gets a "✓". If a requirement does *not* to follow from (WADA), it gets an "✗".
- We conclude with our final (and most important) Fundamental Theorem(s). [See Extras for proofs.]

- We assume that "numerical probabilities reflect evidence", *i.e.*, we adopt the following *evidential requirement*.

($\mathcal{R}$) $\succeq$ is representable by some *regular* probability function.

☞ **Fundamental Theorem**. If a comparative confidence relation $\succeq$ satisfies ($\mathcal{R}$), then $\succeq$ satisfies (WADA).

- The proof of our Fundamental Theorem (see Extras) reveals that $\mathcal{I}(\succeq, w)$ is *evidentially proper*, in this sense [13].

Definition (Evidential Propriety). Suppose a judgment set $\mathbf{J}$ of type $\mathfrak{J}$ is supported by the evidence. That is, suppose there exists some evidential probability function $\Pr(\cdot)$ which represents $\mathbf{J}$ (in the appropriate sense of "represents" for judgment sets of type $\mathfrak{J}$). If this is sufficient to ensure that $\mathbf{J}$ minimizes expected inaccuracy (relative to Pr), according to the measure of inaccuracy $\mathfrak{I}(\mathbf{J}, w)$, then we will say that the measure $\mathfrak{I}$ is **evidentially proper**.

☞ Note: the decision to weight $\succ$-mistakes *twice as heavily* as $\sim$-mistakes is *forced* by evidential propriety (see Extras).

Theorem 1. (WADA) entails ($\mathfrak{C}$), *viz.*, (WADA) $\Rightarrow$ (A_1) & (A_2).

Proof.

Suppose $\succeq$ violates (A_1). Because $\succeq$ is total, this means $\succeq$ is such that $\bot \succeq \top$. Consider the relation $\succeq'$ which agrees with $\succeq$ on all comparisons outside the $\langle \bot, \top \rangle$-fragment, but which is such that $\top \succ' \bot$. We have: $(\forall w)\,[\mathfrak{i}(\top \succ' \bot, w) = 0 < 1 \leq \mathfrak{i}(\bot \succeq \top, w)]$. $\square$

Suppose $\succeq$ violates (A_2). Because $\succeq$ is total, this means there is a pair of propositions p and q in $\mathcal{A}$ such that (a) p entails q but (b) $p \succ q$. Consider the relation $\succeq'$ which agrees with $\succeq$ outside of the $\langle p, q \rangle$-fragment, but which is such that $q \succ' p$. The table on the next slide depicts the $\langle p, q \rangle$-fragments of the relations $\succeq$ and $\succeq'$ in the three salient possible worlds w_1–w_3 not ruled out by (a) $p \vDash q$. By (b) & (LO), p and q are not logically equivalent. So, world w_2 is a live possibility, and $\succeq'$ weakly $\mathcal{I}$-dominates $\succeq$. $\square$

w_i	p	q	$\succeq$	$\succeq'$	$\mathcal{I}(\succeq, w_i)$	$\mathcal{I}(\succeq', w_i)$
w_1	T	T	$p \succ q$	$q \succ' p$	0	0
	T	F				
w_2	F	T	$p \succ q$	$q \succ' p$	2	0
w_3	F	F	$p \succ q$	$q \succ' p$	0	0

Theorem 3. (WADA) entails ($\mathfrak{C}_1$).

Proof.

Having already proved Theorem 1, we just need to show that (WADA) entails (A_3). Suppose $\succeq$ violates (A_3). Because $\succeq$ is total, this means there must exist $p, q, r \in \mathcal{A}$ such that (a) $p \vDash q$, (b) $\langle q, r\rangle$ are mutually exclusive, (c) $q \succ p$, but (d) $p \vee r \succeq q \vee r$. Let $\succeq'$ agree with $\succeq$ on every judgment, *except* (d). That is, let $\succeq'$ be such that (e) $q \succ' p$ and (f) $q \vee r \succ' p \vee r$. There are only four worlds (or $\langle p, q, r\rangle$ state descriptions) compatible with the precondition of (A_3). These are the following (state descriptions).

$$w_1 = p \,\&\, q \,\&\, \neg r \qquad w_2 = \neg p \,\&\, q \,\&\, \neg r$$
$$w_3 = \neg p \,\&\, \neg q \,\&\, r \qquad w_4 = \neg p \,\&\, \neg q \,\&\, \neg r$$

By (c) & (LO), p and q are not logically equivalent. As a result, world w_2 is a live possibility. Moreover, (f) will *not* be inaccurate in *any* of these four worlds. But, (d) *must be inaccurate in world* w_2. This suffices to show that $\succeq'$ *weakly* $\mathcal{I}$-dominates $\succeq$. □

Theorem 4. (WADA) does *not* entail ($\mathfrak{C}_2$).

Proof.

Having already proved Theorem 1, we just need to show that (WADA) does *not* entail (A_5). Suppose (a) $\langle p, q\rangle$ and $\langle p, r\rangle$ are mutually exclusive, (b) $q \succ r$, and (c) $p \vee r \succ p \vee q$. It can be shown (by exhaustive search) that *there is no binary relation* $\succeq'$ on the agenda $\langle p, q, r\rangle$ such that (i) $\succeq'$ agrees with $\succeq$ on all judgments *except* (b) and (c), and (ii) $\succeq'$ weakly $\mathcal{I}$-dominates $\succeq$. There are only four alternative judgment sets that need to be compared with $\{(b), (c)\}$, in terms of their $\mathcal{I}$-values across the five possible worlds (w_1–w_5) compatible with the precondition of (A_5): (1) $\{q \sim r, p \vee r \succ p \vee q\}$, (2) $\{r \succ q, p \vee r \succ p \vee q\}$, (3) $\{q \succ r, p \vee r \sim p \vee q\}$, and (4) $\{q \sim r, p \vee r \sim p \vee q\}$. It is easy to verify that none of these alternative judgment sets weakly $\mathcal{I}$-dominates the set $\{(b), (c)\}$, across the five salient possible worlds. Note: this argument actually establishes the *stronger* claim (**Theorem 5**) that (WADA) does *not* entail ($A_5^\star$)/($\mathfrak{C}_2^\star$). □

Fundamental Theorem. If a comparative confidence relation $\succeq$ satisfies ($\mathcal{R}$), then $\succeq$ satisfies (WADA). That is, ($\mathcal{R}$) ⇒ (WADA).

Proof.

Suppose $\Pr(\cdot)$ fully represents $\succeq$. Consider the expected $\mathcal{I}$-inaccuracy, as calculated by $\Pr(\cdot)$, of $\succeq$: $\mathbb{E}\mathcal{I}^{\succeq}_{\Pr} \stackrel{\text{def}}{=} \sum_w \Pr(w) \cdot \mathcal{I}(\succeq, w)$. Since $\mathcal{I}(\succeq, w)$ is a sum of the $\mathfrak{i}(p \succeq q, w)$ for each $\langle p, q\rangle \in \mathcal{A}$, and since $\mathbb{E}$ is linear:

$$\mathbb{E}\mathcal{I}^{\succeq}_{\Pr} = \sum_{p,q \in \mathcal{A}} \mathbb{E}_{\Pr}\mathfrak{i}(p \succeq q, w)$$

(1) Suppose $\Pr(p) > \Pr(q)$. Then we have:
$\mathbb{E}_{\Pr}\mathfrak{i}(p \succ q, w) = 2 \cdot \Pr(q \,\&\, \neg p) < \mathbb{E}_{\Pr}\mathfrak{i}(p \sim q, w) = \Pr(p \not\equiv q)$, *and*
$\mathbb{E}_{\Pr}\mathfrak{i}(p \succ q, w) = 2 \cdot \Pr(q \,\&\, \neg p) < \mathbb{E}_{\Pr}\mathfrak{i}(q \succ p, w) = 2 \cdot \Pr(p \,\&\, \neg q)$.

(2) Suppose $\Pr(p) = \Pr(q)$. Then we have:
$\mathbb{E}_{\Pr}\mathfrak{i}(p \sim q, w) = \Pr(p \not\equiv q) = \mathbb{E}_{\Pr}\mathfrak{i}(p \succ q, w) = 2 \cdot \Pr(q \,\&\, \neg p)$.

As a result, if $\succeq$ is fully representable by *any* $\Pr(\cdot)$, then $\succeq$ cannot be *strictly* $\mathcal{I}$-dominated, *i.e.*, ($\mathfrak{C}_4$) ⇒ (SADA). Moreover, if we assume $\Pr(\cdot)$ to be *regular*, then $\succeq$ must satisfy (WADA) [13]. ∴ ($\mathcal{R}$) ⇒ (WADA). □

Theorem. $a := 2; b := 0$ is *the only* assignment to a, b that ensures the following definition of $\mathfrak{i}$ is *evidentially proper*.

$$\mathfrak{i}(p \succeq q, w) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } q \,\&\, \neg p \text{ is true in } w\text{, and } p \succ q, \\ b & \text{if } q \equiv p \text{ is true in } w\text{, and } p \succ q, \\ 1 & \text{if } p \not\equiv q \text{ is true in } w\text{, and } p \sim q, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{m}_4 = \Pr(p \,\&\, q)$, $\mathfrak{m}_3 = \Pr(\neg p \,\&\, q)$, and $\mathfrak{m}_2 = \Pr(p \,\&\, \neg q)$. Then, the propriety of $\mathfrak{i}$ is equivalent to the following (universal) claim. And, *the only assignment that makes this (universal) claim true is* $a := 2; b := 0$.

$$\mathfrak{m}_2 + \mathfrak{m}_4 > \mathfrak{m}_3 + \mathfrak{m}_4 \Rightarrow \left(\begin{array}{c} a \cdot \mathfrak{m}_3 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \le a \cdot \mathfrak{m}_2 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \\ \& \\ a \cdot \mathfrak{m}_3 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \le \mathfrak{m}_2 + \mathfrak{m}_3 \end{array} \right)$$

$$\&$$

$$\mathfrak{m}_2 + \mathfrak{m}_4 = \mathfrak{m}_3 + \mathfrak{m}_4 \Rightarrow \left(\begin{array}{c} \mathfrak{m}_2 + \mathfrak{m}_3 \le a \cdot \mathfrak{m}_2 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \\ \& \\ \mathfrak{m}_2 + \mathfrak{m}_3 \le a \cdot \mathfrak{m}_3 + b \cdot (1 - (\mathfrak{m}_2 + \mathfrak{m}_3)) \end{array} \right)$$

- Our ordering presuppositions (Totality & Transitivity) are not universally accepted as rational requirements [14, 12, 23].
- In our book [13], we analyze both of the ordering presuppositions in more detail. Specifically, we show that:
  (1) Totality does not follow from weak accuracy dominance avoidance. That is, (WADA) does not entail Totality.
  (2) Transitivity does not from weak accuracy dominance avoidance. That is, (WADA) does not entail Transitivity.
- These two negative results [especially (1)] are probably not very surprising. But, it is somewhat interesting that *none of the three instances of* Transitivity is entailed by (WADA).

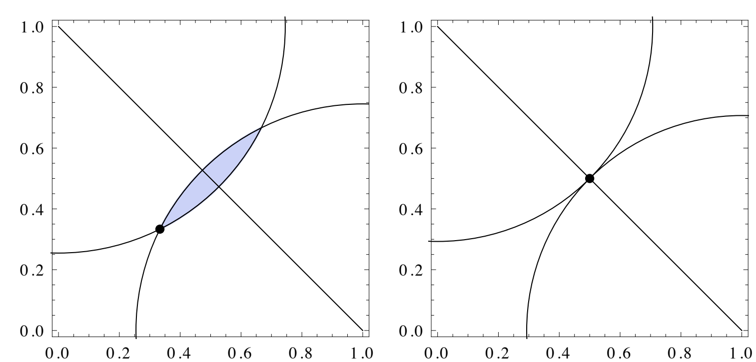
  Transitivity$_1$. If $p \succ q$ and $q \succ r$, then $r \nsucc p$.

  Transitivity$_2$. If $p \succ q$ and $q \sim r$, then $r \nsucc p$.

  Transitivity$_3$. If $p \sim q$ and $q \sim r$, then $p \sim r$.
- The first instance of Transitivity is the *least* controversial of the three. And, the last (transitivity of $\sim$) is the *most* [23].

- In their seminal paper, Kraft *et. al.* [22] refute de Finetti's [3, 4] conjecture: $(\mathfrak{C}_2) \Rightarrow (\mathfrak{C}_4)$. In fact, they show $(\mathfrak{C}_2) \not\Rightarrow (\mathfrak{C}_3)$.
- Their counterexample involves a linear order $\succeq$ on an algebra $\mathcal{B}_{32}$ generated by five states: $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_5\}$.
- We won't write down the entire linear order $\succeq$ as this involves a complete ranking of 32 propositions. Instead, we focus only the following, salient 8-proposition fragment.

$\succeq$	$\mathfrak{s}_1$	$\mathfrak{s}_2 \vee \mathfrak{s}_4$	$\mathfrak{s}_3 \vee \mathfrak{s}_4$	$\mathfrak{s}_1 \vee \mathfrak{s}_2$	$\mathfrak{s}_2 \vee \mathfrak{s}_5$	$\mathfrak{s}_1 \vee \mathfrak{s}_4$	$\mathfrak{s}_1 \vee \mathfrak{s}_2 \vee \mathfrak{s}_4$	$\mathfrak{s}_3 \vee \mathfrak{s}_5$
$\mathfrak{s}_1$	1	1	0	0	0	0	0	0
$\mathfrak{s}_2 \vee \mathfrak{s}_4$	0	1	0	0	0	0	0	0
$\mathfrak{s}_3 \vee \mathfrak{s}_4$	1	1	1	1	0	0	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_2$	1	1	0	1	0	0	0	0
$\mathfrak{s}_2 \vee \mathfrak{s}_5$	1	1	1	1	1	1	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_4$	1	1	1	1	0	1	0	0
$\mathfrak{s}_1 \vee \mathfrak{s}_2 \vee \mathfrak{s}_4$	1	1	1	1	1	1	1	1
$\mathfrak{s}_3 \vee \mathfrak{s}_5$	1	1	1	1	1	1	0	1



- **Simplest case of dF's Theorem [5]**: $b(P) = x$; $b(\neg P) = y$. The diagonal lines are the *probabilistic* b's (on $\langle P, \neg P\rangle$).
- The two directions of de Finetti's theorem (for $\langle P, \neg P\rangle$) can be established *via* these two figures. And, this simplest ($\langle P, \neg P\rangle$) version of the Theorem *generalizes* from the simplest propositional Boolean algebra $\mathcal{B}_4$ to $\mathcal{B}_n$, for *any* n.

- There are two, weaker $\mathcal{I}$-dominance requirements that we discuss in the book [13]. These are as follows.

  Strict Accuracy-Dominance Avoidance (SADA). $\succeq$ should *not be strictly dominated* in inaccuracy (according to $\mathcal{I}$). More formally, there should *not* exist a $\succeq'$ (on $\mathcal{A}$) such that

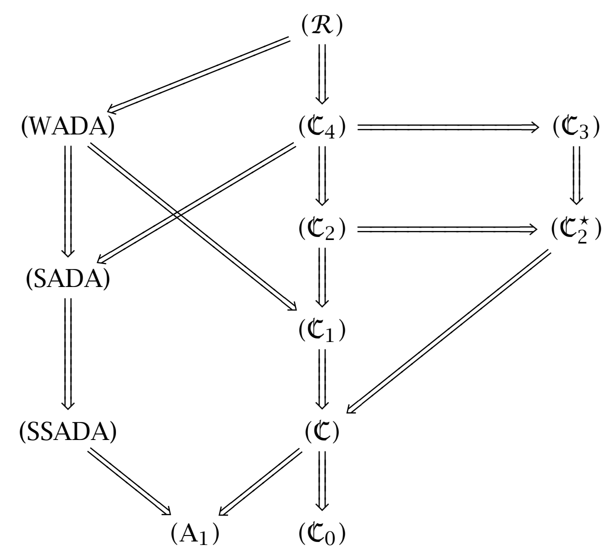
  $$(\forall w)\,[\mathcal{I}(\succeq', w) < \mathcal{I}(\succeq, w)].$$

- Of course, (SADA) is *strictly weaker* than (WADA). And, here is a requirement that is *even weaker* than (SADA).
- Let $\mathbf{M}(\succeq, w) \stackrel{\text{def}}{=}$ the *set* of $\succeq$'s inaccurate judgments at w.

  Strong **Strict Accuracy-Dominance Avoidance** (SSADA). There should *not* exist a $\succeq'$ on $\mathcal{A}$ such that:

  $$(\forall w)\,[\mathbf{M}(\succeq', w) \subset \mathbf{M}(\succeq, w)].$$

- Some of our (WADA) results *also go through for* (SADA) and/or (SSADA). Finally, we give a complete, "big picture" of all the logical relations among all the requirements.



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