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Continuity and completeness of strongly independent preorders[☆]

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HIGHLIGHTS

- Presents conflicts between continuity and completeness assumptions for preorders.
- The preorders are on possibly infinite dimensional convex sets.
- Applications to decision making under risk and uncertainty are given.

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ABSTRACT

We show that a strongly independent preorder on a possibly infinite dimensional convex set that satisfies two of the following conditions must satisfy the third: (i) the Archimedean continuity condition; (ii) mixture continuity; and (iii) comparability under the preorder is an equivalence relation. In addition, if the preorder is nontrivial (has nonempty asymmetric part) and satisfies two of the following conditions, it must satisfy the third: (i′) a modest strengthening of the Archimedean condition; (ii) mixture continuity; and (iii′) completeness. Applications to decision making under conditions of risk and uncertainty are provided, illustrating the relevance of infinite dimensionality.

1. Introduction and main results

The completeness axiom of expected utility has long been regarded as dubious, while the usual continuity axioms are typically seen as innocuous. However, given a strongly independent preorder on a convex set, we show that the standard Archimedean and mixture continuity axioms together imply that the possibilities for incompleteness are highly restricted, in a sense made precise below. In particular, they rule out the most natural preference structures for agents who find they cannot exactly compare two alternatives. If the Archimedean axiom is slightly strengthened in a natural direction, the room for incompleteness vanishes entirely: the preorder must be complete. The first claim strengthens a result of Aumann (1962), while the second extends (Dubra, 2011) from the finite to the infinite dimensional case. We shortly give examples to illustrate the relevance of infinite dimensionality to decision making under risk and under uncertainty.

In more detail, let X be a nonempty convex set, and $\succsim$ a preorder (a reflexive, transitive binary relation) on X. Consider the following axioms. The first is the standard strong independence axiom.

(SI) For $x, y, z \in X$ and $\alpha \in (0, 1)$,

$$x \succsim y \iff \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z.$$

Thus $\succsim$ is an 'SI preorder'. We will be considering the following three Archimedean or continuity axioms.

(Ar) For $x, y, z \in X$, if $x \succ y \succ z$, then $(1-\epsilon)x + \epsilon z \succ y$ for some $\epsilon \in (0, 1)$.

(Ar$^+$) For $x, y, z \in X$, if $x \succ y$, then $(1-\epsilon)x + \epsilon z \succ y$ for some $\epsilon \in (0, 1)$.

(MC) For $x, y, z \in X$, if $\epsilon x + (1-\epsilon)y \succ z$ for all $\epsilon \in (0, 1]$, then $y \succsim z$.

The axiom Ar is weaker than, but for SI preorders equivalent to, the standard Archimedean axiom introduced by Blackwell and Girshick (1954).[1] It is weaker than the axiom Ar$^+$, essentially

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1 See Proposition 5.

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introduced by Aumann (1962).[2] But both Ar and Ar^+ express a similar heuristic. Suppose x is strictly preferred to y, and z is some third alternative. Then Ar says that z cannot be so radically worse than y that a sufficiently small chance of z would disturb the original preference. The axiom Ar^+ extends this by replacing 'worse than' with 'worse than or incomparable with'. For SI preorders, MC is equivalent to the 'mixture-continuity' axiom of Herstein and Milnor (1953), that $\{\alpha \in [0,1] : \alpha x + (1-\alpha)y \succsim z\}$ is closed in $[0,1]$. The displayed formulation is especially normatively natural, and suggests that MC should be seen as just as much an Archimedean condition as Ar and Ar^+.

The final two axioms restrict the possibilities for incomparability. Say that two members x and y of X are *comparable* if $x \succsim y$ or $y \succsim x$. They are *incomparable* if they are not comparable. They have a common upper bound if $z \succsim x$ and $z \succsim y$ for some $z \in X$, and similarly for common lower bound. Recall that an equivalence relation is a binary relation that is reflexive, symmetric, and transitive. The next axiom is nonstandard, while the last is the standard completeness axiom.

(Eq) Comparability is an equivalence relation.

(C) All members of X are comparable.

Using transitivity of $\succsim$, Eq is easily seen to be equivalent to the less technically convenient, but more illuminating

(**Eq**′) For all $x, y \in X$, if x and y are $\succsim$-incomparable, they have neither a common $\succsim$-upper bound, nor a common $\succsim$-lower bound.

This condition does not rule out incomparability. However, in realistic cases where an agent finds it hard to compare two alternatives, she will find it easy to imagine either an alternative that she finds superior to both, or an alternative she finds inferior to both, in each case violating Eq′. Thus while C excludes all incomparability, Eq excludes it in most cases of practical interest.

To state our main result, say that $\succsim$ is *nontrivial* if it has a nonempty strict part; that is, for some $x, y \in X$, $x \succ y$.

Theorem 1. *For any SI preorder $\succsim$ on a convex set X:*
(1) Any two of the following imply the third: MC, Ar and Eq.
(2) For nontrivial $\succsim$, any two of the following imply the third: MC, Ar^+, and C.

The following examples show that various strengthenings of this result are unavailable.

Example 2. (a) Let $X = \mathbb{R}^2$. Set $x \succsim y \Leftrightarrow x_1 \geq y_1$ and $x_2 = y_2$. Then $\succsim$ is nontrivial, MC, Ar, and Eq hold, but Ar^+ and C fail. Thus in (1), Ar cannot be replaced by Ar^+, and Eq cannot be replaced by C.

(b) Let X contain at least two elements. Set $x \succsim y \Leftrightarrow x = y$. Then $\succsim$ is trivial, MC and Ar^+ hold, but C fails. Thus in (2), 'nontrivial' cannot be dropped, and by (a), Ar^+ cannot be replaced by Ar.

Theorem 1 has several precedents. In a seemingly overlooked observation, Aumann (1962) claimed without proof that MC and Ar^+ imply Eq. Thus both parts of the theorem strengthen his claim. In Corollary 11 we strengthen another of Aumann's claims concerning the special case in which X is a vector space.

In the case where X is the set of probability functions on a given finite set, and thus can be identified with the standard simplex of a finite dimensional vector space, the second part of Theorem 1 was proved by Dubra (2011), building on Schmeidler (1971). Dubra's proof makes essential use of finite dimensionality. But placing no restrictions on the dimension of X allows for considerably broader applications,[3] including general sets of probability measures.

Schmeidler's result was that if a nontrivial preorder on a connected topological set has closed weak upper and lower contour sets, and open strict upper and lower contour sets, it must be complete. The axioms we discuss are purely algebraic, making them applicable to cases in which X is not naturally equipped with a topology. Our proof of Theorem 1 is purely algebraic. Indeed the main technical tool, stated in Theorem 6, states equivalences between the three continuity conditions and conditions involving algebraic openness or closedness in ambient vector spaces.

1.1. Discussion

The abstract structure of incomplete SI preorders on convex sets has been discussed, but the relevance of Theorem 1 perhaps has more to do with its compatibility with the typical concrete settings that are used to represent objective risk and subjective uncertainty. To illustrate, let Y be an arbitrary set, Y_c be a compact metric space, and Y_m an arbitrary measurable space; these are typical consequence spaces. Let $P(Y)$ be the set of finitely supported probability measures on Y, $P(Y_c)$ be the set of Borel probability measures on Y_c, and $P(Y_m)$ be an arbitrary convex set of probability measures on Y_m. These are obviously all convex sets, and cover typical cases involving objective risk. Let S_0 and S be a finite and arbitrary sets of states of nature respectively. Then $P(Y)^{S_0}$ is the set of Anscombe–Aumann 'horse lotteries'. Here, members of S_0 are bearers of subjective uncertainty, while the outcomes of horse lotteries are 'roulette lotteries' involving objective risk. For any $x, y \in P(Y)^{S_0}$ and $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in P(Y)^{S_0}$ is defined by setting $(\alpha x + (1-\alpha)y)(s) = \alpha x(s) + (1-\alpha)y(s)$ for any $s \in S_0$, making $P(Y)^{S_0}$ a convex set. Finally, the set Y^S is the set of Savage-acts associating states of nature with consequences; states of nature continue to be the bearers of subjective uncertainty, but no objective risk is modeled. The space Y^S is not naturally a convex set, but given a preorder on Y^S that satisfies reasonably modest axioms, Y^S can be endowed with convex structure; see for example Ghirardato et al. (2003). The importance of allowing X to be infinite dimensional can be seen from the fact that none of these typical domains can be identified with a finite dimensional X. There are many works discussing incomplete SI preorders in the settings just mentioned. Some focus only on SI strict partial orders,[4] but our results are still relevant, as every SI strict partial order is the asymmetric part of some SI preorder.

Given Theorem 6, it is natural to think of Ar and Ar^+ as 'open' conditions, and MC as a 'closed' condition. Both styles of condition have been used extensively in discussions of incomplete SI preorders on the kinds of convex sets just described. In almost every case we know of,[5] the open conditions are at least as strong as Ar in the given model, and the closed conditions are at least as strong as, and typically much stronger than, MC.[6] Thus Theorem 1 has considerable relevance.

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[2] The axiom Aumann actually discusses is $\epsilon_0 x + (1-\epsilon_0)z \succ y \Rightarrow \epsilon x + (1-\epsilon)z \succ y$ for all ϵ close enough to ϵ_0, but for SI preorders, this is equivalent to Ar^+.

[3] Recall that the dimension of a convex set X is the dimension of $\mathrm{Span}(X - X)$, or, equivalently, the dimension of the smallest affine space containing X.

[4] A strict partial order is a binary relation that is transitive, irreflexive, and asymmetric.

[5] The exceptions are Aumann (1962), who imposes a continuity condition that is strictly weaker than both Ar and MC, and Seidenfeld et al. (1995) who impose a similar condition in the Anscombe–Aumann setting.

[6] For open conditions, see Bewley (2002), Manzini and Mariotti (2008), Galaabaatar and Karni (2012, 2013), Evren (2014) and McCarthy et al. (2017c). For closed conditions, see Shapley and Baucells (1998), Ghirardato et al. (2003), Dubra et al. (2004), Nau (2006), Baucells and Shapley (2008), Evren (2008), Kopylov (2009), Gilboa et al. (2010), Danan et al. (2012), Ok et al. (2012) and McCarthy et al. (2017a). Without any continuity condition, one faces incomplete analogues of the situation analyzed by Hausner and Wendel (1952); see Borie (2016), Hara et al. (2016) and McCarthy et al. (2017b).

Although continuity conditions are often said to be technical assumptions, they express natural normative ideas. The meaning of Ar is transparent, while both Ar^+ and MC express different ways of ruling out infinitesimal value differences. Since that basic idea is so widely accepted in discussions of complete SI preorders (when all three continuity conditions are equivalent), it is rather remarkable that ruling out infinitesimal value differences across the board forces one to accept the heavily criticized completeness axiom. Thus we suggest that Theorem 1 deserves to be seen as an impossibility result. We end by canvassing some possible responses.

First, one could try to assess the abstract plausibility of the two styles of continuity conditions, open and closed. There are various possibilities. One could try to argue that one is more plausible than the other. Alternatively, one could think, for example, that Ar^+ and MC are comparably plausible (Aumann, 1962; Manzini and Mariotti, 2008), or that they are difficult to compare in terms of plausibility. The latter positions find no clear difference in terms of plausibility, so someone who finds completeness implausible may be unwilling to accept either style of condition (Aumann, 1962). In any case, one might think that the plausibility of the conditions depends on the concrete interpretation in question. Second, for applications, one could try to develop mirror theories for open and closed conditions, and analyze the sensitivity of their implications to these conditions. Third, one could choose between the two styles of conditions on the basis of the convenience of the representation theorems they support (compare Evren, 2014). Fourth, to reconcile the two forms of continuity with incompleteness, one could adopt a nonstandard model of the relationship between strict partial orders and associated preorders (Karni, 2011; Galaabaatar and Karni, 2012). Fifth, one could argue that in some settings, the case for both styles of condition is strong enough that it provides a novel normative argument for completeness (in a different context, compare Broome, 1999), or even a new argument against strong independence.

2. Proofs

2.1. Preliminaries

Throughout X is a nonempty convex set. We assume X is given to us as a subset of a vector space V (otherwise set $V = \text{Span}\, X$).[7] The following provides a useful representation of the subspace $\text{Span}(X - X)$.

Lemma 3. *Let X be a nonempty convex subset of a vector space V. Then*

$$\text{Span}(X - X) = \{\lambda(x - x') : x, x' \in X, \lambda > 0\}.$$

Proof. The right-hand side is clearly included in the left. For the converse, let $v \in \text{Span}(X - X)$. The case $v = 0$ is trivial, so let $v = \sum_{i=1}^n \lambda_i (x_i - x_i')$ with $x_i, x_i' \in X$, $\lambda_i \neq 0$ for all i, and $n \in \mathbb{N}$. Exchange x_i with x_i' if necessary to have each $\lambda_i > 0$. Set $\lambda = \sum_{i=1}^n \lambda_i$, $x = \frac{1}{\lambda}\sum_{i=1}^n \lambda_i x_i$, and $x' = \frac{1}{\lambda}\sum_{i=1}^n \lambda_i x_i'$. Then $x, x' \in X$ by convexity, and $v = \lambda(x - x')$ as needed. □

Recall that a vector preorder $\succsim_V$ is a preorder on a vector space V such that for any $v, w, u \in V$ and $\alpha > 0$, $v \succsim_V w$ implies $\alpha v + u \succsim_V \alpha w + u$. We define $\{\succsim_V 0\} := \{v \in V : v \succsim_V 0\}$, and similarly $\{\succ_V 0\}$, $\{\sim_V 0\}$. We also define $\precsim$, $\precsim_V$ and sets such as $\{\precsim_V 0\}$ in the obvious way; for example, $x \precsim y \Leftrightarrow y \succsim x$.

[7] By Hausner (1954, Thm. 3.2), a mixture space can be seen as a convex subset of a vector space, so our results apply to arbitrary mixture spaces. We thank a referee for reminding us of this.

Proposition 4. *Let $\succsim$ be a SI preorder on a nonempty convex subset X of a vector space V. For any $v, w \in V$, define $\succsim_V$ by*

$$v \succsim_V w \iff v - w = \lambda(x - y) \text{ for some } x, y \in X, \lambda > 0 \text{ with } x \succsim y.$$

Then $\succsim_V$ is a vector preorder on V, and $\succsim$ is its restriction to $X \times X$. Moreover, $\succsim$ is complete if and only if $\succsim_V$ is complete on $\text{Span}(X - X) = \text{Span}\{\succsim_V 0\}$. Clearly $\text{Span}\{\succsim_V 0\} \subset \text{Span}(X - X)$.

Proof. Clearly $\succsim_V$ is reflexive. Suppose $u \succsim_V v$ and $v \succsim_V w$. Then for some $\lambda, \mu > 0$ and $x_1, x_2, y_1, y_2 \in X$, we have $u - v = \lambda(x_1 - x_2)$, $v - w = \mu(y_1 - y_2)$, $x_1 \succsim x_2$ and $y_1 \succsim y_2$. The former imply $u - w = (\lambda + \mu)(\frac{\lambda x_1 + \mu y_1}{\lambda + \mu} - \frac{\lambda x_2 + \mu y_2}{\lambda + \mu})$; the latter and applications of SI imply $\frac{\lambda x_1 + \mu y_1}{\lambda + \mu} \succsim \frac{\lambda x_2 + \mu y_2}{\lambda + \mu}$. It follows that $u \succsim_V w$, establishing transitivity of $\succsim_V$, so $\succsim_V$ is a preorder on V.

Suppose $v \succsim_V w$, $u \in V$, $\alpha > 0$. For some $\lambda > 0$, $x, y \in X$, we have $v - w = \lambda(x - y)$ with $x \succsim y$. Then $(\alpha v + u) - (\alpha w + u) = \alpha\lambda(x - y)$. This implies $\alpha v + u \succsim_V \alpha w + u$, so $\succsim_V$ is a vector preorder.

Clearly $x \succsim y$ implies $x \succsim_V y$. Conversely, suppose $x \succsim_V y$ for some $x, y \in X$. Then for some $x', y' \in X$, $\lambda > 0$, $x - y = \lambda(x' - y')$ with $x' \succsim y'$. The former implies $\alpha x + (1 - \alpha) y' = \alpha y + (1 - \alpha) x'$ where $\alpha := \frac{1}{1 + \lambda}$; the latter and SI imply $\alpha x + (1 - \alpha) x' \succsim \alpha x + (1 - \alpha) y'$. Substituting, then using SI again, yields $x \succsim y$, hence $\succsim$ is the restriction of $\succsim_V$.

The completeness claim follows from Lemma 3. □

We henceforth assume that $(X, \succsim)$ and $(V, \succsim_V)$ are as in Proposition 4. The fact that $\succsim_V$ is a vector preorder is used repeatedly in the sequel.

2.2. Algebraic conditions

The following assembles facts about the Archimedean conditions.

Proposition 5.
(a) Ar is equivalent to each of the following two conditions.

$$\text{For all } x, y, z \in X : x \succ y \text{ and } y \succ z \Longrightarrow (1 - \epsilon)x + \epsilon z \succ y \text{ and } y \succ \epsilon x + (1 - \epsilon) z \text{ for some } \epsilon \in (0, 1). \tag{1}$$

$$\text{For all } v, w \in V : v, w \succ_V 0 \Longrightarrow v \succ_V \epsilon w \text{ for some } \epsilon > 0. \tag{2}$$

(b) If one of Ar, (1), (2) and Ar^+ holds for some ϵ_0, it holds for all $\epsilon \in [0, \epsilon_0]$.
(c) Ar^+ implies Ar (but not conversely).

Proof. (a) To show (2) ⇒(1), assume (2) and suppose $x \succ y \succ z$. Set $x_\alpha := (1 - \alpha)(x - y) + \alpha(z - y)\ \forall \alpha \in [0, 1]$. Then $x_0 \succ_V 0 \succ_V x_1$, and from (2) one deduces that $(1 - \epsilon)x_0 \succ_V \epsilon(-x_1)$ for small $\epsilon > 0$, i.e. $(1 - \epsilon)(x - y) \succ_V \epsilon(y - z)$, implying $(1 - \epsilon)x + \epsilon z \succ y$ for some $\epsilon \in (0, 1)$. From (2) one also deduces $(1 - \epsilon)(-x_1) \succ_V \epsilon x_0$ for small $\epsilon > 0$, hence $y \succ \epsilon x + (1 - \epsilon) z$ for some $\epsilon \in (0, 1)$. This establishes (1), and clearly (1) ⇒ Ar.

To show Ar ⇒(2), assume Ar and suppose $v, w \succ_V 0$. By Proposition 4, $v = \lambda(x - y)$, $w = \mu(s - t)$ for some $x \succ y$, $s \succ t$, $\lambda, \mu > 0$. By SI, $\frac{1}{2}(x + s) \succ \frac{1}{2}(y + s) \succ \frac{1}{2}(y + t)$. By Ar, for some $\epsilon \in (0, 1)$, $(1 - \epsilon)\frac{1}{2}(x + s) + \epsilon\frac{1}{2}(y + t) \succ \frac{1}{2}(y + s)$. This implies $(1 - \epsilon)(x - y) \succ_V \epsilon(s - t)$, hence $v \succ_V \frac{\epsilon}{1 - \epsilon} \cdot \frac{\lambda}{\mu} w$, establishing (2).

(b) Assume Ar^+, and suppose $x \succ y$ and $(1 - \epsilon_0)x + \epsilon_0 z \succ y$ for some $\epsilon_0 > 0$. These and SI imply $(1 - \epsilon)x + \epsilon z \succ y$ for all $\epsilon \in [0, \epsilon_0]$. The claims about Ar and (1) are proved similarly, and the claim about (2) is clear.

(c) That Ar^+ implies Ar is obvious. The failure of the converse is shown by Example 2(a). □

Recall that a subset S of a vector space W is *algebraically open in* W if for all $v \in S, w \in W, v+\epsilon w \in S$ for all sufficiently small $\epsilon > 0$. S is *algebraically closed in* W if for all $v, w \in W$: $(1-\alpha)v + \alpha w \in S$ for all $\alpha \in (0, 1] \Rightarrow v \in S$. Given $v, w \in W$, we sometimes write $[v, w) \subset W$ for the line segment $\{(1-\alpha)v + \alpha w : \alpha \in [0, 1)\}$. Then S is algebraically closed if $w \in S$ whenever $[v, w) \subset S$. Observe that if S is algebraically closed in W, and S is a subset of another vector space W', then S is algebraically closed in W'. Thus if S is algebraically closed in some vector space, we say that S is *algebraically closed*. The following connects these algebraic notions with our continuity axioms.

Theorem 6.
(a) Ar holds if and only if $\{\succ_V 0\}$ *is algebraically open in* Span$\{\succ_V 0\}$.[8]

(b) Ar$^+$ *holds if and only if* $\{\succ_V 0\}$ *is algebraically open in* Span$(X-X)$.
(c) MC holds if and only if $\{\succsim_V 0\}$ *is algebraically closed.*

Proof. (a) Suppose $\succsim$ satisfies Ar. Let $v \in \{\succ_V 0\}, w \in$ Span$\{\succ_V 0\}$. Clearly we can write $w = a - b$ where each of a and b is either 0 or in $\{\succ_V 0\}$. Since $\succsim_V$ is a vector preorder, $v + \epsilon_1 a \succ_V 0$ for all $\epsilon_1 > 0$. By Proposition 5(a)(2), we have $v \succ_V \epsilon_2 b$ for all sufficiently small $\epsilon_2 > 0$. These imply $v + \epsilon w \succ_V 0$ for all small enough $\epsilon > 0$. This shows that $\{\succ_V 0\}$ is algebraically open in Span$\{\succ_V 0\}$.

Conversely, suppose $\{\succ_V 0\}$ is algebraically open in Span$\{\succ_V 0\}$, and that $v, w \succ_V 0$. Since $-w \in$ Span$\{\succ_V 0\}$, $v + \epsilon(-w) \in \{\succ_V 0\}$ for all sufficiently small $\epsilon > 0$. By Proposition 5, $\succsim$ satisfies Ar.

(b) Assume Ar$^+$. Let $c \in \{\succ_V 0\}, v \in$ Span$(X-X)$. By Lemma 3 and Proposition 4, $c = \alpha(x-y), v = \beta(p-q)$ for some $x, y, p, q \in X$ with $x \succ y, \alpha, \beta > 0$. Then SI, Ar$^+$ and Proposition 5(b) imply $(1-\epsilon)\frac{1}{2}(x+q)+\epsilon\frac{1}{2}(x+p) \succ \frac{1}{2}(y+q)$, hence $(x-y)+\epsilon(p-q) \succ_V 0$, for all sufficiently small $\epsilon > 0$. Consequently $c + \epsilon\frac{\alpha}{\beta}v \succ_V 0$ for small enough $\epsilon > 0$.

Conversely, suppose $\{\succ_V 0\} \cap$ Span$(X-X)$ is algebraically open in Span$(X-X)$. Suppose $x \succ y$ and $z \in X$. Then $x - y \in \{\succ_V 0\}$ and $z - y \in$ Span$(X-X)$, hence for some $\epsilon \in (0, 1)$, $(1-\epsilon)(x-y) + \epsilon(z-y) \in \{\succ_V 0\}$, implying $(1-\epsilon)x + \epsilon z \succ y$.

(c) This follows easily from Shapley and Baucells (1998, Lemma 1.5); for convenience we give a shorter proof.

Suppose MC holds. Suppose that for some $v, w \in$ Span$\{\succsim_V 0\}$, $(1-\alpha)v + \alpha w \in \{\succsim_V 0\}$ for all $\alpha \in (0, 1]$. To show that $\{\succsim_V 0\}$ is algebraically closed, it is sufficient to show $v \in \{\succsim_V 0\}$. For $\alpha \in \mathbb{R}$, define $w_\alpha := (1-\alpha)v + \alpha w$. Suppose first $w_\alpha \sim_V 0$ for some $\alpha \in (0, 1]$. Fix $\beta \in (0, \alpha)$, and let $\kappa = \frac{\alpha}{\alpha-\beta}$. This implies $v = (1-\kappa)w_\alpha + \kappa w_\beta$. Since $w_\alpha \sim_V 0$, $w_\beta \succsim_V 0$, and $\kappa > 0$, both terms are in $\{\succsim_V 0\}$, hence so is v. We are therefore reduced to the case where $w_\alpha \succ_V 0$ for all $\alpha \in (0, 1]$ $(*)$. By Proposition 4, $v, w \in$ Span$(X-X)$, so we may write $v = \lambda_1(x_1-y_1)$ and $w = \lambda_2(x_2-y_2)$ as in Lemma 3. Since for all $\alpha \in [0, 1]$, $w_\alpha \succsim_V 0 \Leftrightarrow \lambda w_\alpha \succsim_V 0$ for any $\lambda > 0$, by a common rescaling of v and w, we may assume $\lambda_1 + \lambda_2 = 1/2$. Let $y = (y_1+y_2)/2$. This implies $v + y = \lambda_1 x_1 + \lambda_1 y_2 + \lambda_2 y_1 + \lambda_2 y_2 \in X$, since X is convex and the coefficients are positive and sum to 1. Similarly, $w + y \in X$. By $(*)$, $(1-\alpha)(v+y) + \alpha(w+y) = w_\alpha + y \succ_V y$ for all $\alpha \in (0, 1]$. Both the left-hand and right-hand sides of that expression are in X, so by Proposition 4, $(1-\alpha)(v+y)+\alpha(w+y) \succ y$ for $\alpha \in (0, 1]$. By MC, $v+y \succsim y$, and by Proposition 4 again, $v+y \succsim_V y$, so $v \in \{\succsim_V 0\}$.

Conversely, suppose $\{\succsim_V 0\}$ is algebraically closed, and that for some $x, y, z \in X$, $\alpha x+(1-\alpha)y \succ z$ for $\alpha \in (0, 1]$. By Proposition 4, for $\alpha \in (0, 1]$, $\alpha x+(1-\alpha)y \succ_V z$, hence $\alpha(x-z)+(1-\alpha)(y-z) \succ_V 0$. Algebraic closure implies $y - z \succsim_V 0$, so by Proposition 4, $y \succsim z$, establishing MC. □

[8] The latter condition is often phrased as '$\{\succ_V 0\}$ is relatively algebraically open.' For elaboration on the algebraic concepts, see e.g. Ok (2007).

Corollary 7. *$\precsim$ satisfies MC if and only if $\succsim$ satisfies MC.*

Proof. The set $\{\precsim_V 0\} = -\{\succsim_V 0\}$ is algebraically closed if and only if $\{\succsim_V 0\}$ is, so Theorem 6(c) yields the equivalence. □

2.3. Proof of Theorem 1

Define $x \bowtie y \Leftrightarrow x \succsim y$ or $y \succsim x$; that is, x and y are $\succsim$-comparable.

Lemma 8. *Let $x_i \bowtie y_i, \alpha_i \in \mathbb{R}$ $(i = 1, \ldots, n)$. Then $\sum_{i=1}^{n}\alpha_i(x_i-y_i) = \alpha(p-q)$, where $\alpha > 0$ and $p \bowtie z \bowtie q$ for some $z \in X$.*

Proof. 1° *Case* $n = 2$. Without loss of generality, assume $\alpha_2, \alpha_1 > 0$; $\alpha_2 \geq \alpha_1$; and $\alpha_2 = 1$. Set $p_k = x_k, q_2 = y_2, q_1 = (1-\alpha_1)x_1+\alpha_1 y_1$ to have $\sum_{i=1}^{2}\alpha_i(x_i-y_i) = p_1 - q_1 + p_2 - q_2$. Clearly $x_1 \bowtie q_1$. Set $p := \frac{1}{2}(p_1+p_2)$, $q := \frac{1}{2}(q_1+q_2)$, $z := \frac{1}{2}(q_1+p_2)$, to have $\sum_{i=1}^{2}\alpha_i(x_i-y_i) = 2(p-q)$ and $p \bowtie z \bowtie q$.

2° *General case*. Without loss of generality, assume $x_i \succsim y_i$ for all i. If $\alpha_1, \alpha_2 > 0$, then

$$\alpha_1(x_1-y_1) + \alpha_2(x_2-y_2) = \alpha'(x_1'-y_1')$$

where $\alpha' = \alpha_1 + \alpha_2 > 0$, $x_1' = (1-\alpha'')x_1 + \alpha'' x_2$, $y_1' = (1-\alpha'')y_1 + \alpha'' y_2$, $\alpha'' = \alpha_2/(\alpha_1+\alpha_2)$, and hence $x_1' \succsim y_1'$, by SI. This way, by induction, we combine all terms having $\alpha_i > 0$. If all the α_i are strictly positive (or similarly, strictly negative), the result is immediate. Otherwise, similarly combine the terms with $\alpha_i < 0$, then apply 1° to the two. □

Lemma 9. *Let W be any vector space, and $S \subset W$. If S is algebraically open in W, then $W \setminus S$ is algebraically closed. The converse holds if S is convex.*

Proof. The claim is Ok (2007, Exercise G.1.5.30). The first claim is clear. For the converse, suppose S is not algebraically open in W. Then for some $v \in S, w \in W$, $\{(1-\alpha)v + \alpha w : \alpha \in [0, \epsilon]\} \not\subset S$ for all $\epsilon > 0$. If S is convex, this implies $\{(1-\alpha)v + \alpha w : \alpha \in (0, \epsilon_0]\} \subset W \setminus S$ for some $\epsilon_0 > 0$. If $W \setminus S$ is algebraically closed, we have $v \in W \setminus S$, a contradiction. □

Recall that $\succsim$ is nontrivial if $\succ \neq \varnothing$.

Proposition 10.
(a) *The following three conditions are equivalent.*

(i) *Both Ar and MC hold.*
(ii) $\{\succsim_V 0\} = \{\sim_V 0\} + [0, \infty)c$ *for some* $c \succsim_V 0$.
(iii) *Ar holds and $\succsim_V$ is complete on* Span$\{\succsim_V 0\}$.

(b) *$\succsim_V$ is complete on* Span$\{\succsim_V 0\}$ *if and only if $\bowtie$ is an equivalence relation.*
(c) *If $\succsim$ is nontrivial, then both* Ar$^+$ *and MC hold if and only if $\succsim$ is complete and Ar holds.*
(d) *If $\succsim$ is nontrivial, then* Span$\{\succ_V 0\}$ = Span$\{\succsim_V 0\}$.

Proof. (a) We show (ii) ⇔ (i) ⇔ (iii). We obtain (ii) ⇒ (i) by using Proposition 5(a)(2) for Ar and by Theorem 6(c) for MC. Conversely, assume (i). If $\succ_V = \varnothing$, (ii) is immediate, so pick $c \succ_V 0$. Suppose for a contradiction $\{\succsim_V 0\} \neq \{\sim_V 0\} + [0, \infty)c =: Q$. Clearly Q is contained in $\{\succsim_V 0\}$, so there exists $d \succ_V 0$ such that $d \notin Q$. By Ar and Theorem 6(a), since $\{\succ_V 0\}$ is algebraically open in Span$\{\succ_V 0\}$, $\alpha(-c) + (1-\alpha)d \succ_V 0$ for sufficiently small $\alpha > 0$. Since $\succsim_V$ is a vector preorder, the set of such α is an interval and is bounded above by 1. Let α_0 be its supremum, and set $e := \alpha_0(-c)+(1-\alpha_0)d$. By MC and Theorem 6(c), $e \succsim_V 0$, and hence $\alpha_0 \in (0, 1)$. By Ar and Theorem 6(a), $e \not\succ_V 0$. Hence $e \sim_V 0$, implying $(1-\alpha_0)d \sim_V e + \alpha_0 c \in Q$, a contradiction. Thus (ii) $\Leftrightarrow$ (i).

Assume (i). Since (i) implies (ii), we have $\{\succsim_V 0\} = \{\sim_V 0\} + [0, \infty)c$ and $\text{Span}\{\succsim_V 0\} = \{\sim_V 0\} + \mathbb{R}c$ for some $c \succsim_V 0$, so $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$, establishing (iii). Conversely, assume (iii), and for a contradiction suppose MC does not hold. Since $\{\succsim_V 0\}$ is a subset of the vector space $\text{Span}\{\succsim_V 0\}$, by Theorem 6(c), there is some $[a, b) \subset \{\succsim_V 0\}$ with $b \in \text{Span}\{\succsim_V 0\}$ and $b \not\succsim_V 0$. By completeness of $\succsim_V$ on $\text{Span}\{\succsim_V 0\}$, $-b \succ_V 0$. This is a contradiction, since Ar and Theorem 6(a) imply $\{\succ_V 0\}$ is algebraically open in $\text{Span}\{\succsim_V 0\}$, but $[-a, -b) \subset \{\precsim_V 0\}$. Hence (i) $\Leftrightarrow$ (iii).

(b) Suppose $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$. Then $x \bowtie y$ and $y \bowtie z \Rightarrow x - y, y - z \in \pm\{\succsim_V 0\} \subset \text{Span}\{\succsim_V 0\}$, implying $x - z \in \text{Span}\{\succsim_V 0\}$, so $x \bowtie z$, implying $\bowtie$ is transitive, and hence an equivalence relation as it is clearly reflexive and symmetric.

Conversely, suppose $\bowtie$ is an equivalence relation. Let $v \in \text{Span}\{\succsim_V 0\} = \text{Span}\{x - y : x \succsim y\}$, by Proposition 4. Then $v = \sum_{i=1}^{n}\alpha_i(x_i - y_i)$ for some $\alpha_i \in \mathbb{R}$, $x_i \succsim y_i$. By Lemma 8, $v = \alpha(p-q)$, where $\alpha > 0, p, q \in X$ with $p \bowtie z$ and $z \bowtie q$ for some $z \in X$. Transitivity of $\bowtie$ implies $p \bowtie q$, hence $v \bowtie_V 0$, implying that $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$.

(c) Assume $\succsim$ is nontrivial. Suppose $\succsim$ satisfies Ar^+ and MC. By Proposition 5(c), $\succsim$ satisfies Ar. By Theorem 6(b), $\{\succ_V 0\}$ is algebraically open in $\text{Span}(X - X)$. Let $w \in X - X$, and by nontriviality, pick $v \in \{\succ_V 0\}$. Then $v + \epsilon w \in \{\succ_V 0\}$ for sufficiently small $\epsilon > 0$, so $w \in \text{Span}\{\succ_V 0\}$; that is, $X - X \subset \text{Span}\{\succ_V 0\}$. By (a), $\succsim_V$ is complete on $X - X$, hence $\succsim$ is complete.

Conversely, when $\succsim$ is complete and satisfies Ar, it must satisfy Ar^+ by SI. By Proposition 4, $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$, hence by (a), $\succsim$ satisfies MC.

(d) The left-hand side is clearly contained in the right. But for nontrivial $\succsim_V$, $\{\sim_V 0\} \subset \text{Span}\{\succ_V 0\}$. To show this, let $v \in \{\succ_V 0\}$, $w \in \{\sim_V 0\}$; then $v + w \in \{\succ_V 0\}$, hence $w \in \text{Span}\{\succ_V 0\}$. This implies $\text{Span}\{\succsim_V 0\} \subset \text{Span}\{\succ_V 0\}$. $\square$

Proof of Theorem 1. (1) If $\succsim$ is trivial, clearly all three of Ar, MC, and Eq hold, so suppose $\succsim$ is nontrivial. Now if $\succsim$ satisfies Ar and MC, Eq must hold by Proposition 10(a, b).

Assume Eq. Then $\succsim_V$ is complete on $\text{Span}\{\succsim_V 0\}$ by Proposition 10(b). If Ar holds, then so does MC, by Proposition 10(a)(iii, i). Assume MC. Then $\{\precsim_V 0\}$ is algebraically closed, by Corollary 7 and Theorem 6(c), hence $\{\succ_V 0\}$ is algebraically open in $\text{Span}\{\succsim_V 0\}$, by Lemma 9 and completeness on $\text{Span}\{\succsim_V 0\}$. But $\text{Span}\{\succsim_V 0\} = \text{Span}\{\succ_V 0\}$, by Proposition 10(d), hence Ar holds, by Theorem 6(a).

(2) By Proposition 10(c), if Ar^+ and MC hold, then so does C. By Proposition 5(c) and Proposition 10(c), if Ar^+ and C hold, then so does MC. Finally, MC and C imply Ar by Theorem 1(1), and C and Ar imply Ar^+ by SI. $\square$

Remark. Aumann (1962) noted that when $X = V$ is a finite dimensional vector space, MC and Ar^+ imply that X may be written as the direct sum of two subspaces such that two elements are comparable if and only if their second coordinates are identical. The following strengthens this claim, by dropping finite dimensionality and weakening Ar^+ to Ar.

Corollary 11. *Suppose $\succsim$ is an SI preorder satisfying Ar and MC, and that $X = V$ is a vector space. Then $V = V_1 \oplus V_2$, and $v, w \in V$ are comparable if and only if $v_2 = w_2$.*

Proof. By Proposition 10(a), Ar and MC imply $\{\succsim 0\} = \{\sim 0\} + [0, \infty)c$ for some $c \succsim 0$. Set $V_1 = \{\sim 0\} + \mathbb{R}c$. Clearly V_1 is a subspace of V, so write $V = V_1 \oplus V_2$. It is immediate that $v \bowtie w \Leftrightarrow v_2 = w_2$. $\square$

References

Aumann, R., 1962. Utility theory without the completeness axiom. Econometrica 30, 455–462.
Baucells, M., Shapley, L., 2008. Multiperson utility. Games Econom. Behav. 62, 329–347.
Bewley, T., 2002. Knightian decision theory: Part I. Decis. Econ. Finance 25 (2), 79–110.
Blackwell, D., Girshick, M., 1954. Theory of Games and Statistical Decisions. John Wiley, New York.
Borie, D., 2016. Lexicographic expected utility without completeness. Theory and Decision 81, 167–176.
Broome, J., 1999. Is incommensurability vagueness? In: Ethics Out of Economics. Cambridge University Press, Cambridge, pp. 123–161.
Danan, E., Guerdjikova, A., Zimper, A., 2012. Indecisiveness aversion and preference for commitment. Theory and Decision 72, 1–13.
Dubra, J., 2011. Continuity and completeness under risk. Math. Social Sci. 61, 80–81.
Dubra, J., Maccheroni, F., Ok, E., 2004. Expected utility theory without the completeness axiom. J. Econom. Theory 115, 118–133.
Evren, Ö., 2008. On the existence of expected multi-utility representations. Econom. Theory 35, 575–592.
Evren, Ö., 2014. Scalarization methods and expected multi-utility representations. J. Econom. Theory 151, 30–63.
Galaabaatar, T., Karni, E., 2012. Expected multi-utility representations. Math. Social Sci. 64, 242–246.
Galaabaatar, T., Karni, E., 2013. Subjective expected utility with incomplete preferences. Econometrica 81, 255–284.
Ghirardato, P., Maccheroni, F., Marinacci, M., Siniscalchi, M., 2003. A subjective spin on roulette wheels. Econometrica 71, 1897–1908.
Gilboa, I., Maccheroni, F., Marinacci, M., Schmeidler, D., 2010. Objective and subjective rationality in a multiple prior model. Econometrica 78, 755–770.
Hara, K., Ok, E., Riella, G., 2016. Coalitional expected multi-utility theory. Mimeo . http://economics.mit.edu/files/12659.
Hausner, M., 1954. Multidimensional utilities. In: Thrall, R., Coombs, C., Davis, R. (Eds.), Decision Processes. John Wiley, pp. 167–180.
Hausner, M., Wendel, J., 1952. Ordered vector spaces. Proc. Amer. Math. Soc. 3, 977–982.
Herstein, I., Milnor, J., 1953. An axiomatic approach to measurable utility. Econometrica 21, 291–297.
Karni, E., 2011. Continuity, completeness and the definition of weak preferences. Math. Social Sci. 62, 123–125.
Kopylov, I., 2009. Choice deferral and ambiguity aversion. Theor. Econ. 4, 199–225.
Manzini, P., Mariotti, M., 2008. On the representation of incomplete preferences over risky alternatives. Theory and Decision 65, 303–323.
McCarthy, D., Mikkola, K., Thomas, T., 2017a. Representation of strongly independent preorders by sets of scalar-valued functions. MPRA Paper No. 79284, https://mpra.ub.uni-muenchen.de/79284/.
McCarthy, D., Mikkola, K., Thomas, T., 2017b. Representation of strongly independent preorders by vector-valued functions. MPRA Paper No. 80806, https://mpra.ub.uni-muenchen.de/80806/.
McCarthy, D., Mikkola, K., Thomas, T., 2017c. Representation of strongly independent strict partial orders by sets of scalar-valued functions. Mimeo.
Nau, R., 2006. The shape of incomplete preferences. Ann. Statist. 34, 2430–2448.
Ok, E., 2007. Real Analysis with Economic Applications. Princeton University Press, Princeton.
Ok, E., Ortoleva, P., Riella, G., 2012. Incomplete preferences under uncertainty: Indecisiveness in beliefs vs. tastes. Econometrica 80, 1791–1808.
Schmeidler, D., 1971. A condition for the completeness of partial preference relations. Econometrica 39, 403–404.
Seidenfeld, T., Schervish, M., Kadane, J., 1995. A representation of partially ordered preferences. Ann. Statist. 23, 2168–2217.
Shapley, L., Baucells, M., 1998. Multiperson utility. UCLA Working Paper 779.