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Econometrica, Vol. 39, No. 2 (March, 1971)

A CONDITION FOR THE COMPLETENESS OF PARTIAL PREFERENCE RELATIONS

BY DAVID SCHMEIDLER[1]

THEOREM: *Let X be a connected topological space and let $\geqslant$ be a binary relation on X which is* (i) *transitive: for all x, y, z in X, $x \geqslant y$ and $y \geqslant z$ imply $x \geqslant z$;* (ii) *open: for each x in X the sets $\{y|y > x\}$ and $\{y|x > y\}$ are open ($x > y$ means $x \geqslant y$ and not $y \geqslant x$);* (iii) *closed: for each x in X the sets $\{y|y \geqslant x\}$ and $\{y|x \geqslant y\}$ are closed;* and (iv) *non-trivial: there are $\bar{x}, \bar{y}$ in X with $\bar{x} > \bar{y}$. Then the relation $\geqslant$ is complete (for all x, y in X, $x \geqslant y$ or $y > x$).*

A binary relation fulfilling Condition (i) is called a partial preference relation. Conditions (i), (ii), and (iii) are common in mathematical analysis of economic equilibrium. Instead of Condition (iv), the much stronger conditions of nonsaturation (i.e., for each x there is y with $y > x$) or desirability, are often assumed, whereas X is assumed to be a convex subset of a linear topological space.

Order properties of preference relations have intuitive meaning in the context of the behavioral sciences. This is not the case with topological conditions. They are only assumed in order to utilize the mathematical tools applied in the analysis of some problems in the behavioral sciences. They may imply, however, as in the theorem, a very restrictive condition of plausible nature.

PROOF: First the following proposition is proved.

(*) *If $x > y$, then $\{z|z > y\} \cup \{z|x > z\} = X$.*

One has

$$\{z|z > y\} \cup \{z|x > z\} \subset \{z|z \geqslant y\} \cup \{z|x \geqslant z\}.$$

We shall show the converse inclusion. If $u \geqslant y$ and not $u > y$, then $y \geqslant u$. Because of the assumption $x > y$ we get $x > u$, i.e., u belongs to the left side of the inclusion above. This proof also covers the case where $x \geqslant u$ because of the symmetry of the relations $\geqslant$ and $\leqslant$. So, we obtain an equality between two sets, one of which is open, by Condition (ii), and the second of which is closed, by Condition (iii). These sets are nonempty because of the assumption $x > y$, hence the connectedness of the space X implies Proposition (*).

Now, assume, per absurdum, that there are v and w in X which are incomparable. Because of Proposition (*) and Condition (iv), $v > \bar{y}$ or $\bar{x} > v$. Without loss of generality let $v > \bar{y}$. Apply Proposition (*) to the pair $v, \bar{y}$; i.e., $w > \bar{y}$ or $v > w$. The relation $v > w$ is impossible by the incomparability assumption so $v > \bar{y}$ and $w > \bar{y}$. We shall prove that $\{z|v > z\} \cap \{z|w > z\}$ is equal to $\{z|v \geqslant z\} \cap \{z|w \geqslant z\}$. The first intersection is nonempty (it contains $\bar{y}$), it is not all of X, and it is an open set. The second intersection is closed. Thus the equality contradicts the connectedness of X.

To prove the equality, let $v \geqslant z$ and $w \geqslant z$. The relation $z \geqslant v$ implies $w \geqslant v$ and the relation $z \geqslant w$ implies $v \geqslant w$, so the only remaining possibility is $v > z$ and $w > z$, which completes the proof.

REMARK: In case of violation of Condition (iv), the preference relation may not be complete.

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However, by defining $x \geqslant^* y$ for every unordered pair x, y in X, we get a complete extension of the original relation. The extended relation satisfies, of course, Conditions (i), (ii), and (iii).

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