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Expected utility theory on mixture spaces without the completeness axiom[☆]

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ABSTRACT

A mixture preorder is a preorder on a mixture space (such as a convex set) that is compatible with the mixing operation. In decision theoretic terms, it satisfies the central expected utility axiom of strong independence. We consider when a mixture preorder has a multi-representation that consists of real-valued, mixture-preserving functions. If it does, it must satisfy the mixture continuity axiom of Herstein and Milnor (1953). Mixture continuity is sufficient for a mixture-preserving multi-representation when the dimension of the mixture space is countable, but not when it is uncountable. Our strongest positive result is that mixture continuity is sufficient in conjunction with a novel axiom we call countable domination, which constrains the order complexity of the mixture preorder in terms of its Archimedean structure. We also consider what happens when the mixture space is given its natural weak topology. Continuity (having closed upper and lower sets) and closedness (having a closed graph) are stronger than mixture continuity. We show that continuity is necessary but not sufficient for a mixture preorder to have a mixture-preserving multi-representation. Closedness is also necessary; we leave it as an open question whether it is sufficient. We end with results concerning the existence of mixture-preserving multi-representations that consist entirely of strictly increasing functions, and a uniqueness result.

1. Introduction

The importance of allowing for incomplete preferences is by now beyond dispute. In the context of expected utility, von Neumann and Morgenstern (1953, p. 630) themselves remarked, of the completeness axiom, that it is "very dubious, whether the idealization of reality which treats this postulate as a valid one, is appropriate or even convenient". In the first systematic treatment of expected utility without the completeness axiom, Aumann (1962, p. 446) wrote that while all the expected utility axioms are descriptively implausible, the completeness axiom alone is "hard to accept even from the normative viewpoint". With normative questions especially in mind, we address the problem of representing incomplete preferences by sets of utility functions.

Following Aumann (1962) and Shapley and Baucells (1998), we suppose that preferences are given by a preorder on a mixture space, in the sense of Hausner (1954). A mixture space is a set M together with a mixing operation, so that for any elements x and y in M and $\alpha \in [0, 1]$, the element $x\alpha y$ of M is understood to be a mixture of x and y in which x is given weight α and y weight $1-\alpha$. We give the standard axiomatization of mixture spaces in Section 2. For now, the best known example involving uncertainty is when M is the set of probability measures on some outcome space, and $x\alpha y$ is taken to be the probability measure $\alpha x+(1-\alpha)y$. More generally, any convex set, and thus any vector space, is a mixture space, with the mixing operation defined by the same formula.

Given a possibly incomplete preorder $\succsim$ on mixture space M, a *multi-representation* is a nonempty set $\mathcal{U}$ of functions $M \to \mathbb{R}$ such that $x \succsim y$ if and only if, for all $u \in \mathcal{U}$, $u(x) \geq u(y)$.[1]

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1 The concept of a multi-representation of a preorder was introduced in Ok (2002), but the general idea goes back much further. In decision theory, Bewley (1986) is perhaps the earliest explicit example. However, in the guise of a

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It is natural to require that the functions u respect the mixing operation. A function $u: M \to M'$ between mixture spaces is *mixture preserving* when $u(x\alpha y) = u(x)\alpha u(y)$. In a multi-representation, as we have defined it, M' is the vector space of real numbers. So the question we consider is under what conditions a preorder $\succsim$ on M has a *mixture-preserving multi-representation*; that is, under what conditions does it satisfy

MR There is a nonempty set $\mathcal{U}$ of mixture-preserving functions $M \to \mathbb{R}$, such that for all $x, y \in M$,

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } u \in \mathcal{U}.$$

It is well known that *any* mixture space is isomorphic to a convex set. Using this fact, our question is mathematically equivalent to the question of when a preorder on a convex set has a multi-representation consisting of affine (or even linear) functionals on the ambient vector space, restricted to the convex set. We will exploit this equivalence in proofs (see Section 4.1), but we follow Mongin (2001) in thinking that mixture spaces are conceptually more fundamental for decision theory. For example, it is often easier to verify that an algebraic structure of interest to decision theorists is a mixture space than to show directly that it is isomorphic to a convex set.

Much of the literature on mixture-preserving multi-representations has focused on specific types of mixture spaces. Besides sets of probability measures (with different possible assumptions about the underlying measurable space), examples include sets of Savage-acts, at least given mild structural assumptions (Ghirardato et al., 2003); Anscombe–Aumann acts; charges (i.e. finitely additive measures); and vector-valued measures representing imprecise probabilities. Mixture-preserving multi-representations themselves come in a variety of forms. In the popular Anscombe–Aumann setting, for example, incomplete preferences may be a matter of incomplete beliefs, incomplete tastes, or both, and multi-representations can reflect these distinctions.[2]

While one could consider these different frameworks one at a time, taking into account their special features, we think it is interesting to consider the unifying question of when one may obtain a mixture-preserving multi-representation of a preorder on an abstract mixture space. This fits with the appealing methodology of assuming as little mathematical structure as possible, and addressing general questions with general tools.

To introduce our main results, let us mention two axioms that must clearly be satisfied for MR to hold, i.e. for the existence of a mixture-preserving multi-representation.[3] First, the preorder must be what we call a 'mixture preorder': it must satisfy what is arguably the central axiom of expected utility theory, strong independence. Strong independence is not in general a natural assumption for preferences on mixture spaces; few people's preferences satisfy it on the simplex whose points denote different proportions of coffee, milk, and sugar. But it is a plausible normative requirement in the examples of mixture spaces introduced above, which all involve uncertainty. Second, it is not hard to show that if a mixture preorder has a mixture-preserving multi-representation, it must satisfy the mixture continuity axiom of Herstein and Milnor (1953).

The result which sets the stage for our discussion, Theorem 2.1, shows, we think rather surprisingly, that mixture continuity is not sufficient for a mixture preorder to have a mixture-preserving multi-representation. However, mixture preorders that satisfy mixture continuity without having a mixture-preserving multi-representation must be rather complicated; for example, Theorem 2.3 shows that they must have uncountably infinite dimension. This raises the question of whether there are normatively natural ways of strengthening or supplementing mixture continuity that do guarantee MR.

Our strongest positive result, Theorem 2.4, shows that, in combination with mixture continuity, an axiom we call 'countable domination' is sufficient for a mixture preorder to satisfy MR. We provide two interpretations of this axiom. First, it is a member of a natural but apparently novel family of decision-theoretic axioms that constrain what we call the 'Archimedean structure' of the preorder. Another axiom in this family is the standard Archimedean axiom, which is much stronger than countable domination. Second, countable domination may be seen as a dimensional restriction on mixture preorders that is much less demanding than the requirement of countable dimension.

Our strongest negative result, Theorem 2.5, considers what happens if we impose a topology on mixture spaces and upgrade mixture continuity to a stronger continuity condition. It notes that any mixture preorder that satisfies MR must be both continuous and closed in the weak topology, understood as the coarsest topology on the mixture space in which the real-valued mixture-preserving functions are continuous. However, more surprisingly, it also shows that being continuous is not sufficient for MR. We leave it as an open question whether being closed is sufficient.

Section 2 states our axioms more formally and presents our main results. Section 2.1 relates them to the most immediately relevant literature, showing how they extend results of Shapley and Baucells (1998) and answer a question posed by Dubra et al. (2004). Section 3 discusses the interpretation of countable domination. Section 4 provides proofs of our main results; it emphasizes the central ideas, appealing to a series of auxiliary results whose proofs we defer to Appendix C. Section 5 refines our results by considering two topics. Section 5.1 presents results concerning the existence of mixture-preserving multi-representations that consist entirely of strictly increasing functions, and relates them (in Section 5.1.1) to results by Aumann (1962), Kannai (1963), Dubra et al. (2004), Evren (2014) and Gorno (2017). Section 5.2 presents a uniqueness result for mixture-preserving multi-representations that is an abstract version of the uniqueness result of Dubra et al. (2004). Appendix A explains the connection between our axioms and other independence and continuity axioms common in the literature. Appendix B provides a geometrical interpretation of our discussion of Archimedean structures. And, as we mentioned, Appendix C contains proofs of the auxiliary results.

It might seem that the problem we are addressing was settled by Shapley and Baucells (1998, Theorem 1.8). In a result that is still quoted, they claimed that a condition they called 'properness' is necessary and sufficient for a mixture-continuous mixture preorder to have a mixture-preserving multi-representation. We discuss properness further in Section 2.1, but, notwithstanding the

single vector-valued function, rather than a family of scalar-valued functions, multi-representations were envisioned but not developed in von Neumann and Morgenstern (1953, pp. 19–20). There is no reason why the general concept of a multi-representation has to stipulate that the codomain is the real numbers. For an example in which it is taken to be a linearly ordered abelian group, see Pivato (2013).

[2] For examples involving incomplete beliefs, see Bewley (1986, 2002) and Ghirardato et al. (2003). For tastes, see Dubra et al. (2004), Eliaz and Ok (2006), Evren (2008, 2014), Gorno (2017), Hara et al. (2019) and Borie (2020). For beliefs and tastes, see Seidenfeld et al. (1995), Nau (2006) and Ok et al. (2012). For closely related examples, see Manzini and Mariotti (2008) (interval-valued representations), Galaabaatar and Karni (2012) and Galaabaatar and Karni (2013) (nonstandard preorders), and Heller (2012) (justifiable choice).

[3] We define these axioms in Section 2. Slightly different versions of the two axioms are common in the literature; we clarify some of the relationships in Appendix A.

importance of their contribution, the necessity part of this claim is mistaken when the mixture space has infinite dimension.[4]

Finally, we acknowledge the centrality to our results of the work of Klee (1953).

2. Main results

A *mixture space* is a nonempty set M together with a *mixing operation* $m: M \times M \times [0, 1] \to M$ that satisfies axioms shortly to be described. As is customary, when the mixing operation is understood, we write $x\alpha y$ for $m(x, y, \alpha)$. The axioms are then: (i) $x\alpha y = y(1-\alpha)x$; (ii) $x\alpha x = x$; (iii) if $x\alpha z = y\alpha z$ for some $\alpha \neq 0$, then $x = y$; and (iv) $x\alpha(y\beta z) = (x\frac{\alpha}{\alpha+\beta-\alpha\beta}y)(\alpha+\beta-\alpha\beta)z$ if α and β are not both zero.[5] These axioms describe abstract features of convex subsets of vector spaces, where the mixing operation is given by $x\alpha y = \alpha x + (1-\alpha)y$. The first three are self-explanatory, and the last is an associativity axiom.

We will need the notion of the dimension of a mixture space. The standard definition (Hausner, 1954) reduces to the case of convex sets (see Section 4.1). However, it is more in the spirit of our focus on mixture spaces to provide a characterization directly in terms of the mixture-space structure. Given a mixture space M, say that $M' \subset M$ is a *mixture subspace* of M if it is a mixture space under the mixing operation inherited from M. For any nonempty $A \subset M$, let $M(A)$ be the smallest mixture subspace of M containing A. Say that A is *mixture independent* if, for any nonempty $A_1, A_2 \subset A$, $A_1 \cap A_2 = \emptyset \implies M(A_1) \cap M(A_2) = \emptyset$.[6] We define the *dimension* of M, written $\dim M$, to be $|A| - 1$ for any maximal mixture-independent subset A. In Section 4.2.1 we show this is well defined and equivalent to the customary definition.

A *mixture preorder* is a preorder $\succsim$ on a mixture space M that is compatible with the mixing operation in that it satisfies the following axiom:

SI For $x, y, z \in M$, and $\alpha \in (0, 1)$, $x \succsim y \iff x\alpha z \succsim y\alpha z$.

A *preordered mixture space* is a pair $(M, \succsim)$ where M is a mixture space and $\succsim$ is a mixture preorder on M.[7] When M is a convex set of probability measures, SI is *strong independence*, arguably the central axiom of expected utility theory.

We are interested in the question: when does a mixture preorder have a mixture-preserving multi-representation?

Consider the following axiom, introduced by Aumann (1962).[8]

MC For $x, y, z \in M$, if $x\alpha y \succ z$ for all $\alpha \in (0, 1]$, then $y \succsim z$.

As Aumann noted, for mixture preorders, MC is equivalent to the well-known *mixture continuity* axiom of Herstein and Milnor (1953), that $\{\alpha \in [0, 1] : x\alpha y \succsim z\}$ and $\{\alpha \in [0, 1] : z \succsim x\alpha y\}$ are closed in $[0, 1]$ for all $x, y, z \in M$.[9]

Our interest in the axiom MC is prompted by the trivial observation, recorded in the following, that MC is necessary for MR. However, to our surprise, MC is not sufficient:

Theorem 2.1. *For any preordered mixture space $(M, \succsim)$,*

$$MR \implies MC,$$

but the implication is not reversible.

The failure of reversibility is in fact quite general.

Theorem 2.2. *Every mixture space of uncountable dimension has a mixture preorder that satisfies MC but violates MR.*

This raises the question: how might MC be strengthened to guarantee a mixture-preserving multi-representation? We will consider a range of conditions that are stronger than MC. Some we will show are sufficient for a mixture-preserving multi-representation, but not necessary. Others are necessary, but not sufficient. We do not know of a nontrivial condition that is necessary and sufficient, but one of our results will suggest a natural candidate.

A first sufficient condition for MR is suggested by Theorem 2.2: we simply strengthen MC by assuming in addition that $\dim M$ is countable. (Recall that *countable* means either finite or countably infinite.)

Theorem 2.3. *For any preordered mixture space $(M, \succsim)$,*

$$MC \ \& \dim M \text{ is countable} \implies MR,$$

but the implication is not reversible.

The assumption of countable dimension in this result is clearly much stronger than necessary. We will give some examples in Section 3: in particular, Example 3.6 provides two simple ways in which a preordered mixture space of countable dimension that satisfies MC, and consequently MR, can be blown up to one of arbitrarily large dimension that still satisfies both MC and MR.

Instead, our weakest sufficient condition involves an apparently novel axiom that we call *countable domination* (CD). We state it now but will discuss its significance at length in Section 3; in short, it strictly weakens the assumption that $\dim M$ is countable, and can also be seen as a much weaker form of the standard Archimedean axiom.

Let $\Gamma_{\succsim} \subset M \times M$ be the graph of the mixture preorder $\succsim$: it consists of pairs (x, y) with $x \succsim y$. For any (x, y) and (s, t) in $\Gamma_{\succsim}$, say that (x, y) *weakly dominates* (s, t) if $x\alpha t \succsim y\alpha s$ for some $\alpha \in (0, 1)$. The relation of weak domination is a preorder on $\Gamma_{\succsim}$ (see Appendix B). A natural interpretation is that when (x, y) weakly dominates (s, t), the (weakly positive) difference in value between s and t is at most finitely many times greater than that between x and y. Our axiom is

CD There is a countable set $D \subset \Gamma_{\succsim}$ such that each $(s, t) \in \Gamma_{\succsim}$ is weakly dominated by some $(x, y) \in D$.

Our strongest positive result is

4 This was noted in Evren (2005, Ex. 3.5). In our Example 3.1, it is instructive to verify that the mixture preorder always has a mixture-preserving multi-representation, but is never proper when $|S|$ is infinite. We are grateful to Eric Danan for discussion.

5 These are a reordering of the axioms given by Hausner (1954). Mixture *sets*, as used for expected utility theory in e.g. Herstein and Milnor (1953) and Fishburn (1970, 1982) are more general. Terminology varies; Mongin (2001) uses 'non-degenerate mixture sets' for what we are calling mixture spaces. In our terminology, despite the greater generality of mixture sets, Mongin recommends focusing on mixture spaces for the development of decision theory.

6 This is analogous to the following characterization of linear independence of a subset B of a vector space: for any $B_1, B_2 \subset B$, $B_1 \cap B_2 = \emptyset \implies \mathrm{span}(B_1) \cap \mathrm{span}(B_2) = \{0\}$.

7 We note in passing that, if $(M, \succsim)$ is a preordered mixture space, then the quotient $M/\sim$ is also naturally a preordered mixture space, and its mixture preorder $\succsim'$ is actually a partial order ($x \sim' y \implies x = y$). Namely, if $[x] \in M/\sim$ denotes the indifference class of $x \in M$, then mixing is defined by $[x]\alpha[y] = [x\alpha y]$, and $\succsim'$ is defined by $[x] \succsim' [y] \iff x \succsim y$. For many purposes it suffices to consider $M/\sim$ rather than M. In particular, it is not hard to see that $\succsim$ satisfies MR if and only if $\succsim'$ does. But we will focus on M itself.

8 However, Aumann (1962) regarded MC as too strong for his purposes, and instead focused on, in our labelling:

Au For $x, y, z \in M$, if $x\alpha y \succ z$ for all $\alpha \in (0, 1]$, then $z \not\succ y$.

This axiom is strong enough to rule out, for example, the lexicographic ordering of the unit square. But as well as being weaker than MC, for mixture preorders, Au is also weaker than the axiom Ar discussed below. We discuss Aumann's results further in Section 5.1.1.

9 See Section 2.1 and Appendix A for further clarification of the connection between MC, the Herstein–Milnor axiom, and the related axiom WCon used by Shapley and Baucells (1998) and Dubra et al. (2004). In particular, we explain in Lemma A.1(iii) why they are all equivalent for mixture preorders.

Theorem 2.4. *For any preordered mixture space* $(M, \succsim)$,

$$MC \,\&\, CD \implies MR,$$

but the implication is not reversible.

Instead of adding to MC a condition such as CD, we might impose a topology on the mixture space, and upgrade MC to a stronger continuity condition.

Given an arbitrary topological space M, we say that a preorder $\succsim$ on M is *continuous* if, for all $x \in M$, the sets $\{y \in M : y \succsim x\}$ and $\{y \in M : x \succsim y\}$ are closed in M. A stronger continuity-like condition that is sometimes used is that the graph $\Gamma_{\succsim}$ is closed in the product topology on $M \times M$; in this case we simply say that $\succsim$ is *closed*.[10] Thus we study the following axioms.

Con $\succsim$ is continuous.

Cl $\succsim$ is closed.

Specific examples of mixture spaces (like sets of probability measures) may suggest specific topologies (see Section 2.1). However, we will focus on what we call the *weak topology*, which makes sense for any mixture space. By definition, it is the coarsest topology (i.e. containing the fewest open sets) such that all the mixture-preserving functions $M \to \mathbb{R}$ are continuous. See Remark 2.6 for more on our terminology. The interest of the weak topology comes from the fact that it makes both Cl and Con into necessary conditions for MR, as the following elaboration of Theorem 2.1 explains.

Theorem 2.5. *For any preordered mixture space* $(M, \succsim)$ *in which* M *has the weak topology,*

$$MR \implies Cl \implies Con \implies MC,$$

but the second and third implications are not reversible.

As before, the displayed implications are easily proved and essentially well known; the novelty lies in the failures of reversibility.[11] In particular, Theorem 2.5 shows that Con is still not sufficient for MR. This is our strongest negative result; it is somewhat delicate because Con, unlike MC, *does* entail MR when, for example, M is a vector space (see Remark 4.13). For us, it is an open question whether Cl and MR are equivalent. Of course, by Theorem 2.4, all four conditions are equivalent when CD holds.

Remark 2.6. A vector space V is a mixture space, so, as we have defined it, the weak topology on V is the coarsest one that makes every mixture-preserving function $V \to \mathbb{R}$ continuous. Equivalently, it is the coarsest topology on V that makes every linear functional on V continuous, since a function on V is mixture preserving if and only if it is affine (i.e. linear plus a constant).

In the vector space case, there are, of course, a variety of weak topologies, each induced by a given subspace of linear functionals. Similarly, there are a variety of weak topologies on mixture spaces, corresponding to subspaces of mixture-preserving functions. But unless otherwise stated, we will not be discussing other weak topologies, hence our use of the term *the* weak topology. Other basic features of the weak topology on a mixture space are noted in Lemma C.2 in Appendix C.

Following discussion of our axiom CD in Section 3, Section 4 presents proofs of the above results, while relegating technical work to Appendix C. Section 5 refines the picture in two ways. First, if $(M, \succsim)$ is a preordered mixture space, we say that a function $f: M \to \mathbb{R}$ is *increasing* if $x \succsim y$ implies $f(x) \geq f(y)$, and *strictly increasing* if, in addition, $x \succ y$ implies $f(x) > f(y)$. A mixture-preserving multi-representation clearly consists of functions that are increasing, but they need not be strictly increasing. Section 5.1 gives results concerning the existence of mixture-preserving multi-representations that contain only strictly increasing functions. Second, Section 5.2 provides a uniqueness result for mixture-preserving multi-representations that is essentially an abstract version of the uniqueness result given by Dubra et al. (2004).

2.1. Related literature

In Section 1 we noted the wide variety of types of mixture spaces, and forms of mixture-preserving multi-representations, that have been discussed. While it would be desirable to consider whether our abstract results have applications in all of those areas, that project lies well beyond the scope of this article. Instead, we will first discuss how our results improve on those of Shapley and Baucells (1998), and then present one application: we explain how one of our results solves a problem left open by the influential work of Dubra et al. (2004).

Our basic objects of study are preorders on mixture spaces that satisfy SI and MC. It is common – and is done so specifically by Shapley and Baucells, and Dubra et al.– to focus on a slightly different set of basic axioms; we refer to these as 'independence' (Ind), which is strictly weaker than SI, and 'weak continuity' (WCon), which is strictly stronger than MC. However, our axioms SI and MC are together equivalent to their axioms Ind and WCon. This equivalence seems to have been known already by Shapley and Baucells (see their note 1), but since formal statements and proofs are hard to find, we provide details in Appendix A. For ease of comparison, we take the liberty of presenting their results in terms of our axioms and terminology.

Shapley and Baucells used a standard embedding theorem to show that any mixture preorder is naturally associated with an essentially unique convex cone. We explain this technique, which we will also use, in Section 4.1. They called a mixture preorder 'proper' if its cone has a nonempty relative algebraic interior; see Section 4.2.4 for the definition. Their main result on mixture-preserving multi-representations showed that every proper mixture preorder that satisfies MC also satisfies MR. As Shapley and Baucells observed, properness holds automatically when the mixture space is finite-dimensional. Thus they effectively proved a weaker version of our Theorem 2.3, in which 'countable' is replaced by 'finite'. More importantly, our Theorem 2.4 strengthens their main result, as our axiom CD is much weaker than their assumption of properness. Indeed, properness is equivalent to a strengthening of CD that we call 'singleton domination' (SD), to be introduced in Section 3.

The assumption of properness was criticized by Dubra et al. (2004, p. 127): "Unfortunately, it is not at all easy to see what sort of a primitive axiom on a preference relation would support such a technical requirement". Our axioms CD and SD are not subject to this kind of criticism. They are formulated directly in terms of the preorder, and, as we explain in Section 3, they are members of a natural family of axioms that place limits on the complexity of the preorder in terms of its Archimedean structure. The standard Archimedean axiom is a much stronger axiom of this type.

[10] In the study of arbitrary preorders on topological spaces, the distinction between these two forms of continuity is standard, but terminology varies. For example, Evren and Ok (2011) use 'semicontinuous' and 'continuous' for our 'continuous' and 'closed' respectively. Bosi and Herden (2016) use 'semi-closed' and 'closed'.

[11] Given the displayed implications (which in fact hold for any reasonable topology (see Lemma 4.10)), and our interest in obtaining MR from natural conditions, one might ask what the interest is of the negative results about MC and Con; why not just focus on Cl? A standard answer is that it is of interest to find the weakest natural condition that suffices. We can add that in our experience, Cl is generally much harder to verify with respect to both the weak topology, and many reasonable topologies, than Con and the even simpler MC.

Dubra et al. (2004) consider the mixture space $M = P(X)$ of Borel probability measures on a compact metric space X. Let $C(X)$ be the set of continuous functions $X \to \mathbb{R}$. They endow $P(X)$ with the narrow topology (or what Dubra et al. call the topology of weak convergence): the coarsest topology such that all the functions $P(X) \to \mathbb{R}$, defined by integrating against functions in $C(X)$, are continuous.[12] Their expected multi-utility theorem shows that Cl is enough to ensure that any mixture preorder on M has a mixture-preserving multi-representation that consists of expectational functions: functions of the form $p \mapsto \int_X u \, dp$ for some $u \in C(X)$.[13] They raise the question of whether this result would hold if Cl was weakened to Con or MC, noting only that MC is enough when X is a finite set.[14] Our Theorem 2.2 shows that Cl *cannot* be weakened to MC in their expected multi-utility theorem, since when X is infinite, $P(X)$ has uncountable dimension. We do not know whether Cl can be weakened to Con in their result, but Theorem 2.5 (see further Remark 4.13) shows that there can be no general inference from Con to Cl.[15]

There is large body of literature on the general question of when a preorder on an arbitrary topological space has a continuous multi-representation (a condition we call CMR). In requiring a mixing-structure, along with mixture-preserving multi-representations, the focus of this article has been different. In the general setting, it is well-known that being closed is not sufficient for CMR. As far as we know, the strongest necessary condition for CMR to hold is given by Bosi and Herden (2016), under the assumption that the topology is first countable. Bosi and Herden remark that they do not see any possibility for satisfactorily avoiding that assumption. Turning back to our setting, the weakest sufficient condition we have for MR to hold is the conjunction of MC and another type of countability condition, CD. Despite the fact that CD is clearly a long way from necessary for MR, we likewise do not see a satisfactory strategy for weakening it.

3. Countable domination

We now discuss our axiom CD, and provide some examples. First, we show that it is a natural weakening of the well-known Archimedean axiom, and connect it with the idea of Archimedean classes. Second, we discuss the extent to which it weakens the assumption that M has countable dimension.

3.1. Countable domination as a weak Archimedean axiom

To better understand CD, we now introduce two more axioms that are in the same natural class. As we will explain, the axioms in this class can be interpreted as constraining the order complexity of mixture preorders.

Given a preordered mixture space $(M, \succsim)$, let $\Gamma_\succ \subset \Gamma_\succsim$ consists of pairs (x, y) with $x \succ y$. Our first axiom is the following.

Ar Every $(x, y) \in \Gamma_\succ$ weakly dominates every $(s, t) \in \Gamma_\succsim$.

It is not hard to show that, for mixture preorders, Ar is equivalent to the *Archimedean axiom* as stated by von Neumann and Morgenstern (1953): if $x \succ y$ and $y \succ z$, then $x\alpha z \succ y$ and $y \succ x\beta z$ for some α and β in $(0, 1)$. We give a proof in Appendix A, Lemma A.2.

Our second axiom is notable because of its close connection to the approach of Shapley and Baucells (1998); see Section 2.1. We call this apparently novel axiom *singleton domination.*

SD There is some $(x, y) \in \Gamma_\succsim$ that weakly dominates every $(s, t) \in \Gamma_\succsim$.

Both of these axioms are stronger than CD:

$$\text{Ar} \implies \text{SD} \implies \text{CD}.$$

The first implication is trivial when $\Gamma_\succ$ is nonempty. When it is empty, both Ar and SD hold automatically, in the latter case because every (x, x) in $\Gamma_\succsim$ weakly dominates every (s, t) in $\Gamma_\succsim$. For the second implication, notice that SD is the special case of CD when D is a singleton. The implications, however, are irreversible, as shown by the next example. Further examples contrasting Ar, SD and CD will be given below.

Example 3.1. Let S be a non-empty set, and $M = \mathbb{R}^S_0$, the vector space of finitely-supported functions $S \to \mathbb{R}$. As a vector space, it is also a mixture space. Define a mixture preorder on M by

$$f \succsim g \iff f(s) \geq g(s) \text{ for all } s \in S.$$

Then $\succsim$ satisfies MC. It satisfies Ar if and only if $|S| = 1$. It satisfies SD if and only if $|S|$ is finite; it satisfies CD if and only if $|S|$ is countable. To illustrate when $|S|$ is countable, define $D = \{(1_A, 0) : A \subset S, A \text{ finite}\}$, where $1_A \in M$ is the characteristic function of A. Then D is a countable subset of $\Gamma_\succsim$, and each $(f, g) \in \Gamma_\succsim$ is weakly dominated by the element $(1_{\text{supp}(f-g)}, 0)$ of D.

The axioms Ar, SD, and CD can also be reformulated in terms of 'Archimedean classes', an idea usually developed in the context of ordered groups or vector spaces (see e.g. Hausner and Wendel, 1952). In the present context of preordered mixture spaces, let us say two pairs (x, y) and (s, t) in $\Gamma_\succsim$ are in the same *Archimedean class* if each weakly dominates the other (this is an equivalence relation, since weak domination is a preorder). Write $[(x, y)]$ for the Archimedean class of (x, y), and let $\Pi_\succsim$ be the set of Archimedean classes in $\Gamma_\succsim$. What we call the *Archimedean structure* of a mixture preorder $\succsim$ is the partially ordered set $(\Pi_\succsim, \geq)$ where $[(x, y)] \geq [(s, t)]$ if and only if (s, t) weakly dominates (x, y).[16] Note that $\Pi_\succsim$ always contains a maximal element, the single Archimedean class consisting of all pairs (x, y) with $x \sim y$. As the following easily proved equivalences show, Ar, SD, and CD can all be seen as placing limits on the complexity of the Archimedean structure.

(a) $\succsim$ satisfies Ar if and only if $(\Pi_\succsim, \geq)$ has at most two elements.
(b) $\succsim$ satisfies SD if and only if $(\Pi_\succsim, \geq)$ contains a minimum element.
(c) $\succsim$ satisfies CD if and only if $(\Pi_\succsim, \geq)$ contains a countable coinitial subset.[17]

12 When X is finite, the narrow topology is equal to what we have called the weak topology; when X is infinite, it is more coarse, i.e. contains fewer open sets, strengthening Con and Cl. As well as by Dubra et al., this strengthened form of Cl is used in the context of multi-representations by e.g. Ghirardato et al. (2003), Ok et al. (2012) and Gorno (2017).

13 Their result contains more detail than this. For discussion and further elaboration, see Evren (2008) and Hara et al. (2019).

14 This follows from the result about finite dimensionality due to Shapley and Baucells (1998) noted above, since every $P(X)$ with X finite is a finite-dimensional mixture space (of dimension $|X| - 1$). The Shapley and Baucells result is slightly stronger though, as not every finite-dimensional mixture space is isomorphic to some $P(X)$. For example, $(0, 1)$ is a one-dimensional mixture space but it is not isomorphic to $P(\{0, 1\}) \cong [0, 1]$.

15 In addition, our Example 4.14 can be modified to show that their result does not hold when Cl is weakened to Con *and* M is allowed to be an arbitrary convex subset of $P(X)$.

16 The direction of the inequality may be surprising, but it is standard in the related literature on valuation theory, and may be thought of as saying that (s, t) comes earlier in order of importance than (x, y).

17 Recall that a subset S' of a preordered set $(S, \succsim_S)$ is *coinitial* if and only if, for every $s \in S$, there exists $s' \in S'$ with $s \succsim_S s'$.

Specifically, if CD holds with respect to a countable $D \subset \Gamma_{\succsim}$, then $\{[(x, y)] : (x, y) \in D\}$ is a countable coinitial subset of $\Pi_{\succsim}$.

There are of course many other ways of limiting the complexity of Archimedean structures, but these are the ones of immediate interest. Appendix B provides more formal discussion of Archimedean structures; here we illustrate with some examples.

Example 3.2. In Example 3.1, for $(f, g), (h, k) \in \Gamma_{\succsim}$, (f, g) weakly dominates (h, k) if and only if $\operatorname{supp}(f - g) \supset \operatorname{supp}(h - k)$. Therefore $[(f, g)] \mapsto \operatorname{supp}(f - g)$ is an isomorphism between the Archimedean structure $(\Pi_{\succsim}, \geq)$ and the set of finite subsets of S, partially ordered by $\subset$. The results of Appendix B yield a different description. Consider the convex cone of positive functions,[18] $C = \{f \in \mathbb{R}^S_0 : f \geq 0\}$. For each finite $A \subset S$, $F_A = \{f \in C : \operatorname{supp}(f) \subset A\}$ is a face of C. If $f \in C$, then $F_{\operatorname{supp}(f)}$ is the smallest face containing f; in the terminology of Appendix B, this means that the faces of the form F_A, with A finite, are the *regular* faces of C. Clearly $A \subset B \iff F_A \subset F_B$. So we conclude that $(\Pi_{\succsim}, \geq)$ is isomorphic to the set of regular faces of C, partially ordered by $\subset$. Proposition B.1 generalizes this description. It also notes that $(\Pi_{\succsim}, \geq)$ has at most one minimal element, corresponding to the largest (thus $\subset$-minimal) face C, if it is regular. In the present example, it has a minimal element only if S is finite.

The following example of a lexicographically ordered vector space makes the structure of $(\Pi_{\succsim}, \geq)$ particularly clear (but MC is usually not satisfied).

Example 3.3. Let $(S, \geq)$ be an ordered set, and as in Example 3.1, let $M = \mathbb{R}^S_0$ be the set of finitely supported functions $S \to \mathbb{R}$. For distinct f and g in M, let $s(f, g) = \min\{s \in S : f(s) \neq g(s)\}$. Define a mixture preorder on M by

$$f \succsim g \iff \text{either } f = g, \text{ or } f(s(f, g)) \geq g(s(f, g)).$$

Let $\Pi_{\succ} \subset \Pi_{\succsim}$ be the set of Archimedean classes of *strictly* positive pairs, i.e. the $[(f, g)]$ with $f \succ g$. It merely omits the maximal element of $\Pi_{\succsim}$. One can then see that $[(f, g)] \mapsto s(f, g)$ is an isomorphism of ordered sets between $\Pi_{\succ}$ and S. Thus Ar holds if and only if $|S| \leq 1$; SD holds if and only if S contains a minimal element, e.g. if $S = \mathbb{N}$; and CD holds if and only if S contains a countable coinitial subset, e.g. if $S = \mathbb{R}$.

Remark 3.4. Most of our examples in this section concern vector spaces. However, this is only for simplicity. Indeed, if $(M, \succsim)$ is a preordered mixture space (a vector space or otherwise), and M' is any mixture space of the same dimension, then there is a mixture preorder on M' with the same Archimedean structure as $\succsim$, and which satisfies MC or MR if and only if $\succsim$ does. (This follows from Propositions B.1(iii) and 4.1.)

3.2. Countable domination and countable dimension

As already mentioned, CD strictly weakens the requirement that the dimension of M be countable; we now discuss how far it relaxes that requirement.

We prove the following in Appendix C.

Proposition 3.5. *If a preordered mixture space has countable dimension, then it satisfies CD. The converse does not hold, even for mixture preorders that satisfy MC.*

We first illustrate the second statement in Proposition 3.5; that is, we illustrate why CD does not entail countable dimensionality, even given MC. One reason is that the dimension of a preordered mixture space can always be increased by introducing extra dimensions of indifference or incomparability, without affecting MC or CD, as the following example shows. Given a preorder $\succsim$, we write $x \curlywedge y$ to indicate incomparability between x and y: neither $x \succsim y$ nor $y \succsim x$.

Example 3.6. Let $(M_1, \succsim_1)$ and $(M_2, \succsim_2)$ be preordered mixture spaces. Consider the preordered mixture space $(M, \succsim)$, in which M is the product $M = M_1 \times M_2$, with the mixture operation defined component-wise, and $\succsim$ is the product preorder: $(x_1, x_2) \succsim (y_1, y_2)$ if and only if $x_1 \succsim_1 y_1$ and $x_2 \succsim_2 y_2$. Suppose that M_1 has countable dimension and that $\succsim_1$ satisfies MC and therefore (by Theorem 2.3 and Proposition 3.5) CD and MR. Consider two cases:

(a) $\succsim_2$ is complete indifference: $x \sim_2 y$ for all $x, y \in M_2$;
(b) $\succsim_2$ is complete incomparability: $x \curlywedge_2 y$ for all distinct $x, y \in M_2$.

In both cases, it is easy to check that $\succsim$, like $\succsim_1$, satisfies MC, CD, and MR. But $\dim M = \dim M_1 + \dim M_2$, which may be arbitrarily high.

Suppose that M_2, and therefore M, has uncountable dimension. Even so, there is a sense in which the preorder $\succsim$ in Example 3.6 only 'cares about' the countably many dimensions contained in M_1. Making this precise suggests that CD might entail a more refined dimensional restriction, such as one of the following.

(A) The quotient mixture space $M/{\sim}$ (cf. note 7) has countable dimension.
(B) For each $x \in M$, the upper and lower sets $\{y \in M : y \succsim x\}$ and $\{y \in M : x \succsim y\}$ (themselves mixture spaces) have countable dimension.

In case (a), the first of these holds, but the second does not: in case (b), the second holds, but not the first. However, in the next example, MC and CD (and hence MR) continue to hold even without (A) or (B).[19]

Example 3.7. Let W be a nontrivial normed vector space, and $M = W \times \mathbb{R}$: as a vector space, M is also a mixture space. Define a mixture preorder on M by

$$(v, a) \succsim (w, b) \iff |v - w| \leq a - b.$$

Then $\succsim$ satisfies MC, SD, and hence CD, but not Ar. To verify CD, take $D = \{(0, 1; 0, 0)\}$. In this case, however, $\dim M = \dim W + 1$, which can be arbitrarily large. Moreover, there is no nontrivial indifference ($x \sim y \implies x = y$), and the upper and lower sets have the same, perhaps uncountable, dimension as M.

For a similar example in which MC and CD hold, but SD (and hence Ar) does not, take the M just described and let $M' = M^{\mathbb{N}}_0$, the set of finitely supported functions $\mathbb{N} \to M$. Define a mixture preorder $\succsim'$ on M' by $f \succsim' g$ if and only if $f(n) \succsim g(n)$ for all n. In analogy to Example 3.1 (in the case of S countably infinite), CD holds for $\succsim'$ with respect to $D = \{(0, 1_A; 0, 0) : A \subset \mathbb{N}, A \text{ finite}\}$.

Nevertheless, despite these examples, there is still a subtle sense in which CD *is* a dimensional restriction; we explain it in Remark 4.8.

Turning to ways in which Proposition 3.5 may be strengthened, it is straightforward to extend Proposition 3.5 to show that *if* (A) holds, then so does CD. One might conjecture that CD holds if (B) holds. However, Example 4.12 provides a counterexample to this conjecture; in it, the upper and lower sets even have finite dimension, and MC holds, but CD and MR do not.

[18] Convex cones are defined and discussed in Section 4.1.

[19] Indeed, not even the quotients of the upper and lower sets by the indifference relation have countable dimension.

4. Proofs of main results and discussion

4.1. From mixture spaces to vector spaces

Our motivation for studying mixture spaces was given in the introduction. However, at a technical level, we will use a standard method to reduce questions about mixture spaces to equivalent, but mathematically more convenient, questions about vector spaces. In the context of multi-representations, this reduction was first used in Shapley and Baucells (1998).[20]

It follows from a standard embedding theorem[21] that any mixture space M can be embedded in a (real) vector space V, in such a way that V is the affine hull of M (or, equivalently, $V = \text{span}(M - M)$), and the mixture operation on M coincides with that on V: $x\alpha y = \alpha x + (1-\alpha)y$. M is, therefore, a convex subset of V, and from this it is easy to show

$$V = \{\lambda(x-y) : \lambda > 0,\ x, y \in M\}. \tag{4.1}$$

We follow Shapley and Baucells in calling such an embedding *efficient*. Efficient embeddings are essentially unique: if $M \subset V$ and $M \subset V'$ are efficient embeddings, then there is a unique affine isomorphism $V \to V'$ that is the identity map on M.

Recall that a *linear preorder* $\succsim_V$ on a vector space V is a preorder on V that is compatible with vector addition and positive scalar multiplication; that is, $v \succsim_V v' \iff \lambda v + w \succsim_V \lambda v' + w$ for all $v, v', w \in V$ and $\lambda > 0$. Let $M \subset V$ be an efficient embedding. (Considering V as a mixture space, a linear preorder is the same as a mixture preorder.) As Shapley and Baucells explain, there are natural one-to-one correspondences between mixture preorders $\succsim$ on M, convex cones $C \subset V$, and linear preorders $\succsim_V$ on V, such that, for all $x, y \in M$,[22]

$$x \succsim y \iff x - y \in C \iff x \succsim_V y. \tag{4.2}$$

This formula explicitly defines $\succsim$ in terms of $\succsim_V$ or C, while the next formulae explicitly define C in terms of $\succsim$, and $\succsim_V$ in terms of C:

$$C = \{\lambda(x-y) : \lambda > 0,\ x \succsim y\} \qquad v \succsim_V 0 \iff v \in C. \tag{4.3}$$

We then call C the *positive cone* of $\succsim$, and $\succsim_V$ the *linear extension* of $\succsim$.

Finally, mixture-preserving functions $u : M \to \mathbb{R}$ correspond one-to-one with affine functions $\tilde{u} : V \to \mathbb{R}$, in such a way that $\tilde{u}$ extends u, that is, $\tilde{u}|_M = u$. Moreover, a set $\mathcal{U}$ of mixture-preserving functions $M \to \mathbb{R}$ is a multi-representation of $\succsim$ if and only if $\{\tilde{u} : u \in \mathcal{U}\}$ is a mixture-preserving multi-representation of $\succsim_V$. It follows from (4.3) that an equivalent condition in terms of C is

$$C = \bigcap_{u \in \mathcal{U}} \{v \in V : \tilde{u}(v) \geq \tilde{u}(0)\}. \tag{4.4}$$

[20] Besides Shapley and Baucells (1998), we refer the reader to Mongin (2001) for a careful study of the embedding it relies on, and to a text such as Ok (2007) for the vectorial concepts.

[21] See Hausner (1954, §3). A more general embedding theorem was given in Stone (1949, Thm. 2), but Hausner's result is easier to apply directly.

[22] Since terminology varies slightly: $C \subset V$ is a convex cone if and only if C is nonempty, convex and $[0, \infty)C = C$. We note that although Shapley and Baucells start with axioms that are different from ours (see Appendix A), they first derive SI from their axioms, then use SI to construct the correspondences we describe here. The correspondence between $\succsim$ and C is stated in their equations (11) and (12); the well-known correspondence between C and $\succsim_V$ follows if we consider V as a mixture space.

4.2. Proofs of main results

We now prove our main results in terms of a series of auxiliary results. We outline the ideas on which the auxiliary results are based, but unless otherwise stated, we defer their full proofs to Appendix C. Given the existence of efficient embeddings, our positive results mainly rely on standard extension and separation techniques in vector spaces. The proofs of the negative results are more striking, and we describe the counterexamples on which they are based.

4.2.1. Preliminaries

Recall that a subset S of a vector space V is *algebraically closed* if $v \in S$ whenever $(v, w] \subset S$. (In standard notation, $(v, w] = \{(1-\alpha)v + \alpha w : \alpha \in (0, 1]\}$.) We say that $S \subset V$ is *weakly closed* in V if it is closed in the weak topology on V (see Remark 2.6). We prove the following proposition in Appendix C.

Proposition 4.1. *Let $(M, \succsim)$ be a preordered mixture space, $M \subset V$ an efficient embedding, and $C \subset V$ the positive cone.*
(i) $\dim M$ equals the vector-space dimension of V.
(ii) $\succsim$ satisfies MC if and only if C is algebraically closed.
(iii) $\succsim$ satisfies MR if and only if C is weakly closed in V.

Part (i) shows that our definition of the dimension of M in Section 2 is equivalent to a more standard characterization (see e.g. Mongin, 2001). Part (ii) is almost the same as Shapley and Baucells (1998, Thm. 1.6), but since our axioms are slightly different, we provide a proof. In fact, we will use (ii) to show that our axioms are equivalent to theirs, in Appendix A. Part (iii) is a routine application of the strong separating hyperplane theorem.

4.2.2. Theorems 2.1 and 2.2

Proof of Theorem 2.1. The proof that MR implies MC is straightforward. Indeed, suppose that $\succsim$ has a mixture-preserving multi-representation $\mathcal{U}$. Suppose given $x, y, z \in M$ such that $x\alpha y \succ z$ for all $\alpha \in (0, 1]$. Then, for any $u \in \mathcal{U}$, $u(x\alpha y) \geq u(z)$. But $u(x\alpha y) = \alpha u(x) + (1-\alpha)u(y)$. In the limit $\alpha \to 0$, we find $u(y) \geq u(z)$. Since this is true for all $u \in \mathcal{U}$, we must have $y \succsim z$, as required for MC.

The fact that the converse fails is immediate from Theorem 2.2, to which we now turn. □

The proof of Theorem 2.2 appeals to the following proposition, further discussed below.

Proposition 4.2. *Let V be a vector space of uncountable dimension. There exists a convex cone in V that is algebraically closed but not weakly closed in V.*

Proof of Theorem 2.2. Let $M \subset V$ be an efficient embedding of a mixture space M of uncountable dimension, so that, by Proposition 4.1(i), V also has uncountable dimension. By Proposition 4.2, V contains a convex cone that is algebraically closed but not weakly closed. Using (4.2), this cone defines a mixture preorder on M. By Proposition 4.1 parts , this mixture preorder satisfies MC but not MR. □

We prove Proposition 4.2 in Appendix C. The proof relies on following example, which is based on Klee (1953). Klee showed that if a vector space has uncountable dimension, then it contains a convex subset that is algebraically closed but not weakly closed (see Köthe (1969, pp. 194–195) for a discussion in more modern terminology). We modify Klee's construction to obtain a convex cone with similar properties.

Example 4.3. Let V be a vector space with an uncountable basis B. Endow V with the weak topology. Given a subset S of V, we write cone(S) for the convex cone in V generated by S, that is, the smallest convex cone that contains S. It consists of all linear combinations of S with non-negative coefficients. Choose $b_0 \in B$, and let $B_1 = B \setminus \{b_0\}$. For each finite, non-empty subset $A \subset B_1$, let $y_A = |A|^{-2} \sum_{b \in A} b$. Define a convex cone

$$K = \text{cone}\,\{y_A + b_0 : A \subset B_1 \text{ is nonempty and finite}\}.$$

The proof of Proposition 4.2 shows that K is algebraically closed but not closed. In fact, this generalizes slightly: the same argument, using separating hyperplanes, shows that K is not closed with respect to *any* locally convex topology on V.

4.2.3. Theorem 2.3

The proof rests on the following, which provides a converse to the result of Klee just mentioned.

Proposition 4.4. *Let V be a vector space of countable dimension. Every convex set in V that is algebraically closed is weakly closed in V.*

This was proved using the algebraic version of the separating hyperplane theorem in Köthe (1969, (3) on p. 194). In Appendix C we provide a slightly different proof: to apply the separating hyperplane theorem, we use a result of Klee (1953), that in a vector space of countable dimension, the finite topology is locally convex.

Proof of Theorem 2.3. Suppose that MC holds and that M has countable dimension. Given an efficient embedding $M \subset V$, V also has countable dimension, by Proposition 4.1(i). By Proposition 4.1(ii), the positive cone C is algebraically closed, so, by Proposition 4.4, it is weakly closed. Therefore, by Proposition 4.1(iii), $\succsim$ satisfies MR.

For a counterexample to the converse implication, let M be an uncountable-dimensional vector space with the preorder of complete indifference: $x \sim y$ for all $x, y \in M$. This satisfies MR despite having uncountable dimension. (Examples 3.6, 3.7 and 4.7 provide less trivial examples.) □

4.2.4. Theorem 2.4

We first interpret CD and, for future reference, SD, in terms of the positive cone. For further discussion of Archimedean structure along similar lines, see Appendix B. Let V be a vector space with linear preorder $\succsim_V$; let C be any subset of V. Recall that the *relative algebraic interior* of C consists of those $v \in C$ with the following property: for every $w \in \text{aff}(C)$, the affine hull of C, there is some $\epsilon > 0$ such that $[v, v + \epsilon w] \subset C$.

Recall also that a set S is *cofinal in* C (with respect to $\succsim_V$) if $S \subset C$ and, for all $v \in C$, there is some $s \in S$ with $s \succsim_V v$.

Proposition 4.5. *Let $(M \succsim)$ be a preordered mixture space, $M \subset V$ an efficient embedding, C the positive cone, and $\succsim_V$ the linear extension.*

(i) $\succsim$ satisfies SD if and only if C has a nonempty relative algebraic interior.
(ii) $\succsim$ satisfies CD if and only if there is a countable set that is cofinal in C.

We will also use the following standard extension theorem, due to Kantorovich (1937). For a proof, see Aliprantis and Tourky (2007, Thm. 1.36).

Theorem 4.6 (*Kantorovich*)**.** *Let V be a vector space with a linear preorder $\succsim_V$. Let W be a linear subspace that is cofinal in V. Then any increasing linear functional on W extends to an increasing linear functional on V.*

Proof of Theorem 2.4. We first give a counter-example to the reverse implication; that is, we give an example of a mixture preorder that satisfies MR (and therefore MC) but not CD.

Example 4.7. Let $M = \mathbb{R}^{\mathbb{N}}$, the vector space of functions $\mathbb{N} \to \mathbb{R}$. Define a mixture preorder on M by $f \succsim g \Leftrightarrow f(n) \geq g(n)$ for all $n \in \mathbb{N}$. This clearly satisfies MR (the canonical projections $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ provide a multi-representation), but it violates CD. *Proof:* In this case, the positive cone C consists of the $f \in M$ with $f(n) \geq 0$ for all n. Suppose that CD holds; by Proposition 4.5(ii), there is a countable subset $\{f_1, f_2, \ldots\}$ cofinal in C. Let $f(k) = f_k(k) + 1 \in C$. Then for no k is it true that $f_k \succsim f$, a contradiction.

Now let $(M, \succsim)$ be a preordered mixture space, satisfying MC and CD; we have to show it satisfies MR. Let $M \subset V$ be an efficient embedding, C the positive cone, and $\succsim_V$ the linear extension. For any subspace $W \subset V$ we let $\succsim_W$ be the restriction of $\succsim_V$ to W, a linear preorder with positive cone $C_W = C \cap W$.

By Proposition 4.5(ii), there is a countable set Z cofinal in C. Given $w \in V \setminus C$, set $Z_w = \text{span}(Z \cup \{w\})$. By Proposition 4.1(ii), C is algebraically closed. It follows that C_{Z_w} is also algebraically closed. Since Z_w has countable dimension, C_{Z_w} is weakly closed in Z_w, by Proposition 4.4. By the strong separating hyperplane theorem (Aliprantis and Border, 2006, Cor. 5.84), there is a linear functional L'_w on Z_w such that $L'_w(C_{Z_w}) \subset [0, \infty)$ and $L'_w(w) < 0$. Because $L'_w(C_{Z_w}) \subset [0, \infty)$, L'_w is an increasing linear functional on Z_w.

Let $Y_w = \text{span}(C \cup \{w\})$. We claim that Z_w is cofinal in Y_w. Indeed, let $y \in Y_w$. We can write it in the form $y = \lambda w + \sum_{c \in C} \lambda_c c$, with $\lambda, \lambda_c \in \mathbb{R}$ and finitely many λ_c being non-zero. Since Z is cofinal in C, we can find, for each $c \in C$, some $z_c \in Z$ with $z_c \succsim_V c$. Since $c \succsim_V 0$, it follows that $|\lambda_c| z_c \succsim_V \lambda_c c$. Therefore $\lambda w + \sum_{c \in C} |\lambda_c| z_c \succsim_V y$. Since the left-hand side of this formula is an element of Z_w, Z_w is cofinal in Y_w.

By Theorem 4.6, L'_w extends from Z_w to an increasing linear functional L''_w on Y_w. Arbitrarily extend L''_w to a linear functional L_w on V. By construction, $L_w(C) \subset [0, \infty)$ and $L_w(w) < 0$. Therefore $C = \bigcap_{w \in V \setminus C} \{v \in V : L_w(v) \geq 0\}$. It follows from (4.4) that $\mathcal{U} = \{L_w|_M : w \in V \setminus C\}$ is a mixture-preserving multi-representation of $\succsim$. □

Remark 4.8. The following variation on Proposition 4.5(ii), also proved in Appendix C, explains the sense in which CD is a dimensional restriction, generalizing the countable dimensionality condition used in Theorem 2.3.

Corollary 4.9. *Let $(M \succsim)$ be a preordered mixture space, $M \subset V$ an efficient embedding, C the positive cone, and $\succsim_V$ the linear extension. Then $\succsim$ satisfies CD if and only if there is a subspace that is cofinal in* span C *and that has countable dimension.*

To illustrate: in Example 3.7, span$(C) = M$, which may have arbitrarily high dimension, but span$\{(0, 1)\}$ is a one-dimensional cofinal subspace.

4.2.5. Theorem 2.5

We begin with a mostly well-known observation that generalizes some of the claims in Theorem 2.5. Say that a preorder $\succsim$ on an arbitrary topological space M has a *continuous multi-representation* if it satisfies

CMR There is a nonempty set $\mathcal{U}$ of continuous functions $M \to \mathbb{R}$, such that for all $x, y \in M$,

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } u \in \mathcal{U}.$$

Lemma 4.10. *Let $\succsim$ be a preorder on a topological space M. Then CMR $\implies$ Cl $\implies$ Con. Moreover, suppose M is a mixture space such that, for each $x, y \in M$, the map $f_{x,y}\colon [0, 1] \to M$ given by $\alpha \mapsto x\alpha y$ is continuous. Then Con $\implies$ MC.*

The proof of Lemma 4.10 is in Appendix C. Here we use it to deduce Theorem 2.5.

Proof of Theorem 2.5. If M is a mixture space with the weak topology, then every mixture-preserving function $M \to \mathbb{R}$ is continuous; therefore MR implies CMR. Moreover, for each $x, y \in M$, the map $f_{x,y}\colon [0, 1] \to M$ given by $\alpha \mapsto x\alpha y$ is continuous. The implications stated in Theorem 2.5 are therefore immediate from Lemma 4.10.

To show that the third implication in Theorem 2.5 cannot be reversed, we need an example that satisfies MC but not Con. We again appeal to Example 4.3. We take $M = V$ and let $\succsim$ be the mixture preorder with positive cone $C = K$. Recall that K is algebraically closed but not weakly closed (as shown in proving Proposition 4.2). By Proposition 4.1(ii), $\succsim$ satisfies MC. Since $K = \{x \in M : x \succsim 0\}$, $\succsim$ violates Con.

Finally, we need to show that Con does not imply Cl. We isolate this claim as the following proposition and prove it separately. □

Proposition 4.11. *There is preordered mixture space $(M, \succsim)$ such that $\succsim$ is continuous but not closed in the weak topology on M.*

The proof of Proposition 4.11, given in Appendix C, involves the following modification of Example 4.3.

Example 4.12. Let V, B, and K be as in Example 4.3. Let $V^+ = \operatorname{cone}(B)$. For any $v \in V^+$, let $A_v \subset B_1$ be the set of elements of B_1 with respect to which v has strictly positive coefficients. Let $V_v = \operatorname{span}(A_v \cup \{b_0\})$, and

$$M = \{(v, w) : v \in V^+, w \in V_v\} \subset V \times V.$$

It is easy to check that this M is a convex set. Let $\succsim$ be the mixture preorder on M with the positive cone $K' = \{(0, w) : w \in K\} \subset V \times V$. That is, for all $(x, y), (v, w) \in M \times M$,

$$(x, y) \succsim (v, w) \iff x - v = 0,\ y - w \in K \cap V_v. \tag{4.5}$$

Equip M with the weak topology. The proof of Proposition 4.11 consists in the verification that $\succsim$ is continuous but not closed.

Remark 4.13. Let $(M, \succsim)$ be a preordered mixture space with the weak topology. As already noted, it follows from Theorems 2.4 and 2.5 that the conditions MR, Cl, Con and MC are equivalent when CD holds. In particular, they are equivalent when $\dim M$ is countable.

In addition, when M is a vector space, the conditions MR, Cl, and Con (but not MC) are equivalent. To show the equivalence, it is sufficient, by Theorem 2.5, to show that Con entails MR. Since M is a vector space, $\succsim$ is a linear preorder, with corresponding positive cone $C = \{x \in M : x \succsim 0\}$. But Con implies that C is closed, implying MR by Proposition 4.1(iii). In fact, the equivalence between Cl and Con holds with respect to any vector topology on the vector space M, because $\Gamma_{\succsim}$ is the preimage of C under the continuous map $(x, y) \mapsto x - y$.

Thus, to obtain a counterexample to "Con $\implies$ Cl", Example 4.12 had to involve a mixture space of uncountable dimension that is not a vector space. However, it is worth noting that there are counterexamples to "Con $\implies$ Cl" involving mixture spaces M with reasonable topologies other than the weak topology; the next example illustrates.

Example 4.14. Let V be the vector space of finitely supported functions $\mathbb{Z} \to \mathbb{R}$, equipped with the topology of pointwise convergence. Here $\mathbb{Z}$ is the set of integers; let $\mathbb{N}$ be the subset of positive integers. Let $M = \{f \in V : \forall n \in \mathbb{N},\ f(n) \geq f(-n) \geq 0\}$. It is a convex subset of V with countably infinite dimension. Give it the topology induced by the topology on V.

For each $n \in \mathbb{N}$, let $b_n = \frac{1}{n}1_{-n} + 1_0$, where $1_{-n} \in V$ is the characteristic function of $\{-n\}$. Let $C = \operatorname{cone}\{b_n : n \in \mathbb{N}\}$. Define a mixture preorder on M by setting $f \succsim g \iff f - g \in C$.

Since $\Gamma_{\succsim} \ni (\frac{1}{n}1_n + b_n, \frac{1}{n}1_n) \to (1_0, 0) \notin \Gamma_{\succsim}$ as $n \to \infty$, $\succsim$ is not closed. To show that $\succsim$ is continuous, suppose given $f \in M$. Let $A = \{n \in \mathbb{N} : f(n) > f(-n)\}$. Let $C_f = \operatorname{cone}\{b_n : n \in A\}$. The convex cone C_f is finitely generated, so by a standard argument, C_f is closed in V. For direct verification though, note that C_f consists of those $g \in V$ such that, for all $n \in \mathbb{Z}$: $g(n) \geq 0$; $g(n) = 0$ unless $n = 0$ or $-n \in A$; and $g(0) = \sum_{n \in A} ng(-n)$. But those are closed conditions.

It is easy to check that $\{g \in M : g \succsim f\} = M \cap (f + C_f)$, and it is therefore closed in M. A similar argument shows that $\{g \in M : f \succsim g\}$ is closed in M, making $\succsim$ continuous.

5. Strict multi-representation and uniqueness

We now briefly discuss two standard topics concerning mixture-preserving multi-representations.

5.1. Strict multi-representation

The pioneering study of expected utility without the completeness axiom of Aumann (1962) focused on the existence of a single real-valued, strictly increasing, mixture-preserving function (as defined in Section 2); see also Fishburn (1982). But such a function does not fully characterize an incomplete preorder, and interest turned to the existence of mixture-preserving multi-representations, which do. One can try to combine these approaches by considering mixture-preserving multi-representations that consist entirely of strictly increasing functions:

SMR There is a nonempty set $\mathcal{U}$ of *strictly increasing* mixture-preserving functions $M \to \mathbb{R}$, such that for all $x, y \in M$,

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } u \in \mathcal{U}.$$

The advantages of such 'strict' multi-representations have been emphasized by Evren (2014) and Gorno (2017), although Evren uses a notion of multi-representation that is different from ours. We now present two basic results about extending MR to SMR. Since our earlier results gave sufficient conditions for MR, results giving sufficient conditions for SMR are implied.

First, we note that if a mixture preorder satisfies MR, then solving Aumann's problem – that is, finding a strictly increasing mixture-preserving function – is enough to guarantee SMR as well.

Proposition 5.1. *Let $(M, \succsim)$ be a preordered mixture space. Then $\succsim$ satisfies SMR if and only if it satisfies MR and there exists a strictly increasing mixture-preserving function $M \to \mathbb{R}$.*

The second result extends the picture given by Theorems 2.2 and 2.3 to representations by strictly increasing functions.

Proposition 5.2. *Let M be a mixture space.*

(i) If $\dim M$ is countable, any mixture preorder on M that satisfies MR also satisfies SMR.

(ii) If $\dim M$ is uncountable, there is a mixture preorder on M that satisfies MR but not SMR.

In common with our earlier results, these results show a sharp difference between the cases of countable and uncountable dimension. Theorem 2.3 and Proposition 5.2 together show that, when $\dim M$ is countable, MC is equivalent to SMR. But when $\dim M$ is uncountable, MC is not sufficient even for MR; and even if MR is satisfied, SMR may not be.

The proof of Proposition 5.1 is very simple. For Proposition 5.2, the main idea of the proof of (i) is that countable dimension enables us to focus on multi-representations with countably many elements, as the following lemma shows. Such a countable multi-representation can be used to construct a strictly increasing function, and Proposition 5.1 applies.

Lemma 5.3. *Let $(M, \succsim)$ be a preordered mixture space. If $\succsim$ has a mixture-preserving multi-representation $\mathcal{U}$, then it has a mixture-preserving multi-representation $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| \le \max(\aleph_0, \dim M)$.*

The proof of Proposition 5.2(ii) rests on the following example.

Example 5.4. Assume that $\dim M$ is uncountable. Let $M \subset V$ be an efficient embedding, so $\dim V$ is uncountable. For some uncountable ordinal κ, we can choose a basis $\{v_\alpha : \alpha < \kappa\} \subset M$ for V indexed by ordinals α smaller than κ. For each $\beta < \kappa$, let π_β be the unique linear functional on V such that $\pi_\beta(v_\alpha) = 1$ if $\alpha = \beta$ and $\pi_\beta(v_\alpha) = 0$ otherwise. For each $\alpha < \kappa$, define a mixture-preserving function u_α on M by $u_\alpha(x) = \sum_{\beta \le \alpha} \pi_\beta(x)$. This is well-defined, since for each x in V, and hence M, $\pi_\beta(x)$ is nonzero for only finitely many β. Let $\mathcal{U} = \{u_\alpha : \alpha < \kappa\}$, and let $\succsim$ be the mixture preorder on M that it represents. The proof of Proposition 5.2(ii) shows that $\succsim$ does not have a strictly increasing function, mixture-preserving or otherwise.

5.1.1. Related literature

Aumann (1962) showed that the condition Au (see note 8) is necessary and sufficient for a mixture preorder on a finite dimensional mixture space to admit a strictly increasing mixture-preserving function. Aumann also showed that this does not hold for mixture spaces of uncountable dimension. In particular, he noted that the partially ordered vector space discussed in Example 4.7 ($\mathbb{R}^{\mathbb{N}}$ with the product partial order) does not admit a strictly increasing mixture-preserving function. He also noted that the dimension of $\mathbb{R}^{\mathbb{N}}$ is $\mathfrak{c}$, the cardinality of the continuum. Aumann does not make these points, but it is clear that his example satisfies MR, and that, using efficient embeddings, it can be generalized to show that any *mixture* space whose dimension is at least $\mathfrak{c}$ admits a mixture preorder that satisfies MR but not SMR. Without assuming the continuum hypothesis, though, this falls short of proving Proposition 5.2(ii), which applies to mixture spaces of *any* uncountable dimension.

Kannai (1963) extended Aumann's results by providing a necessary and sufficient condition for a linear preorder on a countable dimensional vector space to admit a strictly increasing linear functional. This provides an alternative basis for deducing our Proposition 5.2(i). If $(M, \succsim)$ is a preordered mixture space, then, as in Proposition 4.1(iii), MR entails that the positive cone C is closed in the weak topology on V; in particular, its closure is disjoint from $\{v \in V : 0 \succ_V v\}$. Kannai's main result (summarized before his Theorem C) is essentially that, when $\dim V$ is countable, this last condition is necessary and sufficient for the existence of a strictly increasing linear functional on V. Restricted to M, such a functional will be a strictly increasing mixture-preserving function. Our Proposition 5.1 then shows that $\succsim$ satisfies SMR.

As we discussed in Section 2.1, Dubra et al. (2004) consider mixture preorders on the set of probability measures on a compact metric space, and assume Cl with respect to the narrow topology. Besides proving the existence of a mixture-preserving, and indeed expectational, multi-representation, they also prove in their Proposition 3 the existence of a strictly increasing expectational function. Gorno (2017) uses this to prove the existence of a multi-representation by strictly increasing expectational functions. Our proof of Proposition 5.1 is based on a similar technique.

Evren (2014) also considers probability measures on a compact metric space. He does not focus on multi-representations in our sense, but nonetheless gives conditions under which a preorder can be represented by a set of strictly increasing functions in a different sense, which may have some advantages. We note that Evren's approach is essentially incompatible with ours (and with the one of Dubra et al.), insofar as his main continuity axiom, 'open-continuity,' rarely holds when MC does: a mixture preorder that satisfies both is either complete or symmetric.[23]

5.2. Uniqueness

Finally, we give a uniqueness result for mixture-preserving multi-representations. It is very similar to the uniqueness theorem of Dubra et al. (2004), but worked out in our setting of abstract mixture spaces.

Given a mixture space M, we let M^* be the vector space of all real-valued mixture-preserving functions on M. Let $\mathcal{C} \subset M^*$ be the subspace of constant functions. We give M^* the topology of pointwise convergence: the coarsest topology such that for each $x \in M$, the function $M^* \to \mathbb{R}$ given by $f \mapsto f(x)$ is continuous. We write $\overline{S}$ for the closure of a subset S of M^*.

Proposition 5.5. *Let M be a mixture space. Two nonempty sets $\mathcal{U}, \mathcal{U}' \subset M^*$ represent the same preorder on M if and only if $\overline{\mathrm{cone}\,(\mathcal{U} \cup \mathcal{C})} = \overline{\mathrm{cone}\,(\mathcal{U}' \cup \mathcal{C})}$.*

It is easy to check that if $\mathcal{U}$ represents $\succsim$, then the subset of functions in M^* that are increasing with respect to $\succsim$ is the unique *maximal* mixture-preserving multi-representation of $\succsim$. Proposition 5.5 is equivalent to the claim that the closure of the convex cone containing $\mathcal{U}$ and the constant functions *is* this maximal multi-representation.

Appendix A. Comparison of axioms

In this appendix, we clarify the relationship between the axioms we have used and some of the others commonly used in the literature on expected utility theory without completeness.

Let M be a mixture space. We first consider two axioms similar in style to MC:

MC$^+$ For $x, y, z \in M$, $\{\alpha \in [0, 1] : x\alpha y \succsim z\}$ and $\{\alpha \in [0, 1] : z \succsim x\alpha y\}$ are closed.

WCon For $x, y, z, w \in M$, $\{\alpha \in [0, 1] : x\alpha y \succsim z\alpha w\}$ is closed.

Here, MC$^+$ is the version of mixture continuity originally stated as Axiom 2 in Herstein and Milnor (1953); WCon is axiom P4 of Shapley and Baucells (1998), called *weak continuity* by Dubra et al. (2004). As we will show, these axioms are equivalent to MC given SI, but not equivalent given a common weakening of SI, the independence axiom:

Ind For $x, y, z \in M$, and $\alpha \in (0, 1)$, $x \succsim y \implies x\alpha z \succsim y\alpha z$.

[23] Given the existence of efficient embeddings, this follows from McCarthy and Mikkola (2018, Theorem 1(2)); the axiom Ar$^+$ considered there follows from open continuity for any reasonable topology on M. Evren's own Lemma 2 states a version of this result in his particular setting; in the decision-theoretic literature, the general point that there is a tension between 'open' and 'closed' continuity conditions goes back to Aumann (1962, p. 453) and Schmeidler (1971).

Lemma A.1. *Let $\succsim$ be a preorder on a mixture space M.*

(i) WCon $\Longrightarrow$ MC$^+$ $\Longrightarrow$ MC

(ii) WCon & Ind $\not\Longleftarrow$ MC$^+$ & Ind $\not\Longleftarrow$ MC & Ind

(iii) WCon & Ind $\Longleftrightarrow$ MC$^+$ & SI $\Longleftrightarrow$ MC & SI.

One takeaway from Lemma A.1 is that MC is weaker than WCon and SI is stronger than Ind, but, following Shapley and Baucells (1998), we could have focused on the package of Ind and WCon instead of SI and MC. We have emphasized the latter combination partly because MC seems simpler and more intuitive than WCon, and partly because SI is arguably the central idea of expected utility: if M is a convex set of probability measures, SI is necessary and sufficient for a preorder on M to have an vector-valued expectational representation (McCarthy et al., 2020, Lem. 4.3).

Finally, we show that, for mixture preorders, our axiom Ar, stated in Section 3.1, is equivalent to the standard Archimedean axiom of von Neumann and Morgenstern (1953):

Arch For $x, y, z \in M$, $x \succ y \succ z \Longrightarrow$ for some $\alpha, \beta \in (0, 1)$, $x\alpha z \succ y$ and $y \succ x\beta z$.

Lemma A.2. *Let $\succsim$ be a mixture preorder on a mixture space M. Then Arch $\Longleftrightarrow$ Ar.*

Proof of Lemma A.1. (i) MC$^+$ consists of two special cases of WCon, taking $w = z$ or $y = x$. MC follows from the part of MC$^+$ that says that $\{\alpha \in [0, 1] : x\alpha y \succsim z\}$ is closed.

(ii) First take $M = [0, 1]$ and define $\succsim$ by $x \sim y$ if $x = y$ or $x, y \in [0, 1)$, and $x \curlywedge y$ otherwise.[24] This $\succsim$ is easily seen to satisfy MC and Ind but not MC$^+$ (take $x = z = 0$ and $y = 1$). Second, take $M = [0, 1]^2$, and define $\succsim$ so that $(x_1, x_2) \sim (y_1, y_2)$ if $(x_1, x_2) = (y_1, y_2)$ or $x_1 = y_1 \in [0, 1)$, and $(x_1, x_2) \curlywedge (y_1, y_2)$ otherwise. This $\succsim$ is easily seen to satisfy MC$^+$ and Ind but not WCon (take $x = z = (0, 0)$, $y = (1, 0)$, $w = (1, 1)$).

(iii) Lemma 1.2 in Shapley and Baucells (1998) says that WCon & Ind $\Longrightarrow$ SI. The left-to-right implications are then immediate from part (i). For the right-to-left direction, assume MC and SI. SI obviously entails Ind, so it remains to derive WCon. It is possible to give a direct proof, using only the mixture space axioms. However, a shorter proof is available in terms of an efficient embedding. We emphasize that this involves no circularity, as Shapley and Baucells (1998) derived the results concerning efficient embeddings that we presented in Section 4.1 using only SI, having first derived it from WCon and Ind; see note 22.

Assume, then, that $M \subset V$ is an efficient embedding, with C the positive cone. By Proposition 4.1(ii), whose proof does not depend on the present result, C is algebraically closed. Consider the set $I = \{\alpha \in [0, 1] : \alpha x + (1 - \alpha)y \succsim \alpha z + (1 - \alpha)w\}$, as in the statement of WCon. Define $f(\alpha) = \alpha(x - z) + (1 - \alpha)(y - w)$. Thus f maps $[0, 1]$ onto the line segment $I' = [y - w, x - z] = \{\alpha(x - z) + (1 - \alpha)(y - w) : \alpha \in [0, 1]\}$. Since C is convex, $I' \cap C$ is a (possibly empty) line segment; since C is algebraically closed, this line segment, if not empty, contains its end points. By (4.2), $\alpha \in I \iff f(\alpha) \in C$, so $I = f^{-1}(I' \cap C)$. It follows that I is a closed interval, implying WCon. □

Proof of Lemma A.2. Assume SI, so that $\succsim$ is a mixture preorder. A simple consequence is that, for $x, y, s, t \in M$ and $\alpha \in (0, 1)$,

$$\text{If } x \succ y \text{ and } s \succsim t, \text{ or else } x \succsim y \text{ and } s \succ t, \text{ then } x\alpha s \succ y\alpha t. \quad \text{(A.1)}$$

To show that Ar implies Arch, suppose $x \succ y$ and $y \succ z$, so that (x, y) and (y, z) are elements of $\Gamma_{\succ}$. Then, given Ar, (x, y) weakly dominates (y, z): for some $\gamma \in (0, 1)$, $x\gamma z \succsim y\gamma y = y$. For any $\alpha \in (\gamma, 1)$, the mixture space axioms imply $x\alpha z = x\frac{\alpha-\gamma}{1-\gamma}(x\gamma z)$. (The easiest way to check such identities is to reduce to the case in which M is a convex set, using an efficient embedding $M \subset V$.) Since both $x \succ y$ and $x\gamma z \succsim y$, it follows from (A.1) that $x\frac{\alpha-\gamma}{1-\gamma}(x\gamma z) \succ y\frac{\alpha-\gamma}{1-\gamma}y = y$. Therefore $x\alpha z \succ y$, as desired for Arch. We also have that (y, z) weakly dominates (x, y), and a similar argument shows that $y \succ x\beta z$ for some $\beta \in (0, 1)$.

Conversely, suppose given $(x, y) \in \Gamma_{\succ}$ and $(s, t) \in \Gamma_{\succsim}$. If $s \sim t$, then by (A.1), $x\frac{1}{2}t \succ y\frac{1}{2}s$, so (x, y) weakly dominates (s, t) and Ar holds in this case. Suppose instead $s \succ t$. Using (A.1) twice, we first find $x\frac{1}{2}s \succ y\frac{1}{2}s \succ y\frac{1}{2}t$. Arch then implies that $(x\frac{1}{2}s)\alpha(y\frac{1}{2}t) \succ y\frac{1}{2}s$ for some $\alpha \in (0, 1)$. The left-hand side can be rewritten using the mixture-space axioms as $(x\alpha t)\frac{1}{2}(s\alpha y)$, and the right-hand side as $(y\alpha s)\frac{1}{2}(s\alpha y)$. SI then yields $x\alpha t \succsim y\alpha s$, so (x, y) weakly dominates (s, t). □

Appendix B. Weak dominance and archimedean structures

Let $(M, \succsim)$ be a preordered mixture space. In this appendix we prove some general facts about weak dominance that we used in Section 3. Primarily, we show that weak dominance is a preorder on $\Gamma_{\succsim}$. This enables us to define the Archimedean structure $(\Pi_{\succsim}, \geq)$ as in Section 3.1: $\Pi_{\succsim}$ consists of equivalence classes in $\Gamma_{\succsim}$ under the symmetric part of the weak dominance preorder. While it is not difficult to check the preordering property directly, we proceed in a way that highlights a geometrical interpretation of the Archimedean structure: it is closely related to the lattice of faces of the positive cone C defined by an efficient embedding $M \subset V$ (cf. Section 4.1). This was illustrated in Example 3.2.

Recall that a non-empty convex subcone $F \subset C$ is called a *face* of C if, for all $x, y \in C$, $x + y \in F \Longrightarrow x, y \in F$. The set $\mathcal{F}$ of faces is partially ordered by inclusion, and indeed it is a complete lattice.[25] This means in particular that, for any $v \in C$, there is a smallest face $\Phi(v)$ containing v. Let us say that $F \in \mathcal{F}$ is *regular* if F is not the union of its proper subfaces: equivalently, $F = \Phi(v)$ for some $v \in C$. Let $\mathcal{F}_r \subset \mathcal{F}$ be the set of regular faces.

Proposition B.1.

(i) For any $(x, y), (s, t) \in \Gamma_{\succsim}$, (x, y) weakly dominates (s, t) if and only if $\Phi(x - y) \supset \Phi(s - t)$.

(ii) Weak dominance is a preorder on $\Gamma_{\succsim}$.

(iii) $(\Pi_{\succsim}, \geq)$ is isomorphic to $(\mathcal{F}_r, \subset)$ as a partially ordered set.

(iv) Any $[(x, y)] \in \Pi_{\succsim}$ is minimal if and only if $\Phi(x - y) = C$. In particular, $\Pi_{\succsim}$ contains at most one minimal element.

Proof. For (i), suppose that (x, y) weakly dominates (s, t). Then there exists $\alpha \in (0, 1)$ such that $\alpha x + (1 - \alpha)t \succsim \alpha y + (1 - \alpha)s$. Let $\lambda = \frac{1-\alpha}{\alpha}$. It follows from (4.3) that $(x - y) - \lambda(s - t) \in C$. At this point we appeal to Barker (1973, Lemma 2.8): $w \in \Phi(v)$ if and only if there exists $\lambda > 0$ such that $v - \lambda w \in C$. (We note that Barker's lemma does not use his standing assumption of finite-dimensionality.) In our case, we find $s - t \in \Phi(x - y)$, and therefore $\Phi(s - t) \subset \Phi(x - y)$. The argument is reversible.

Part (ii) now follows from the fact that '$\supset$' is a preorder on $\mathcal{F}$.

Now for part (iii). It follows from part (i) that $[(x, y)] \mapsto \Phi(x - y)$ is a well-defined, order-preserving, injective function $\Pi_{\succsim} \to \mathcal{F}_r$, and we just have to show it is surjective, i.e. that every regular face is of the form $\Phi(x - y)$ with $(x, y) \in \Gamma_{\succsim}$. Every regular face is of the form $\Phi(v)$, with $v \in C$, and, by (4.3), every

[24] Recall from before Example 3.6 that $x \curlywedge y$ means that neither $x \succsim y$ nor $y \succsim x$.

[25] See Barker (1973), from which we take our simple definition of a face of a convex cone; it is compatible with the standard definition of the face of a convex set.

such v is of the form $\lambda(x-y)$ with $\lambda > 0$ and $(x,y) \in \Gamma_{\succsim}$. Since every face containing v contains $\frac{1}{\lambda}v$, and vice versa, we find that $\Phi(v) = \Phi(x-y)$.

For (iv), C is the minimal face of C with respect to the preorder '$\subset$' (i.e. it is set-theoretically the *largest* face). So, if $\Phi(x-y) = C$, then certainly C is a minimal regular face, and therefore $[(x,y)]$ is minimal. Conversely, if $[(x,y)]$ is minimal, then $\Phi(x-y)$ is a minimal regular face. It remains to show that, if there is a minimal regular face, then it is C. Suppose $\Phi(v)$ is a minimal regular face. Note that, for any $w \in C$, any face containing $\Phi(\frac{1}{2}v + \frac{1}{2}w)$ contains $v+w$, and therefore contains both v and w. Therefore $\Phi(v) \subset \Phi(\frac{1}{2}v + \frac{1}{2}w)$. Since $\Phi(v)$ is minimal regular, $\Phi(v) = \Phi(\frac{1}{2}v + \frac{1}{2}w) \ni w$. That is, $\Phi(v)$ contains every $w \in C$; so $\Phi(v) = C$. □

Appendix C. Proofs of auxiliary results

Proof of Proposition 3.5. For the first claim, suppose that $(M, \succsim)$ is a preordered mixture space of countable dimension. We appeal to some results from Section 4, the proofs of which do not depend on this one. In the terminology of Section 4.1, let $M \subset V$ be an efficient embedding, with C the positive cone. Proposition 4.1(i) shows that V has countable dimension. Therefore its subspace $\operatorname{span}(C)$ has countable dimension. Corollary 4.9 then tells us that $\succsim$ satisfies CD (note that $\operatorname{span}(C)$ is a cofinal subspace of itself).

The second claim, that the converse does not hold, even for mixture preorders that satisfy MC, is illustrated by Examples 3.6 and 3.7. □

Proof of Proposition 4.1. For (i), let $A \subset M$ be nonempty. Fix any $a_0 \in A$ and let $A' = \{a - a_0 : a \in A \setminus \{a_0\}\}$. Since $M \subset V$ is an efficient embedding, $V = \operatorname{span}(M-M) = \operatorname{span}(M - \{a_0\})$. Thus, A' is a basis for V if and only if it is linearly independent and maximal among linearly independent subsets of $M - \{a_0\}$. We claim that A' is linearly independent if and only if A is mixture independent. It follows that A' is a basis for V if and only if A is a maximal mixture-independent subset of M. Since $|A'| = |A| - 1$, it follows that the vector-space dimension of V equals the mixture-space dimension of M.

To prove the claim, first suppose that A is *not* mixture independent. There must be nonempty $A_1, A_2 \subset A$ such that $A_1 \cap A_2 = \emptyset$ but $M(A_1) \cap M(A_2) \neq \emptyset$. Given the embedding of M into V, $M(A_1)$ equals the convex hull of A_1; it consists of all convex combinations of elements of A_1. Since $M(A_1) \cap M(A_2) \neq \emptyset$, there is an equality between two convex combinations of the form

$$\textstyle\sum_i \alpha_i x_i = \sum_i \beta_i y_i$$

with $\alpha_i, \beta_i \in [0,1]$, $x_i \in A_1$, $y_i \in A_2$, and $\sum \alpha_i = \sum \beta_i = 1$. But then we also have

$$\textstyle\sum_i \alpha_i (x_i - a_0) = \sum_i \beta_i (y_i - a_0)$$

showing that A' is linearly dependent. Conversely, suppose that A' is linearly dependent. Then there are disjoint, finite $A_1, A_2 \subset A \setminus \{a_0\}$ and an equation of the form

$$\textstyle\sum_i \lambda_i (a_i - a_0) = \sum_i \mu_i (b_i - a_0)$$

where at most one of the sums is empty (in which case it is zero), with all $\lambda_i, \mu_i > 0$, $a_i \in A_1$, $b_i \in A_2$. Without loss of generality, we can assume that $\lambda := \sum_i \lambda_i \geq \sum_i \mu_i := \mu$, so that A_1 is nonempty. Moving all terms involving a_0 to the right-hand side, and dividing by λ, we have

$$\textstyle\sum_i \frac{\lambda_i}{\lambda} a_i = \sum_i \frac{\mu_i}{\lambda} b_i + \frac{\lambda-\mu}{\lambda} a_0.$$

This shows that $M(A_1) \cap M(A_2 \cup \{a_0\}) \neq \emptyset$. Therefore A is not mixture independent.

Now for part (ii). Suppose first that $\succsim$ satisfies MC. Let $(v, w] \subset C$, so that, by (4.3), $z \succsim_V 0$ for every $z \in (v,w]$. To show that C is algebraically closed, we have to show $v \in C$. Suppose first that $z \sim_V 0$ for some $z \in (v,w]$. Then $-z \sim_V 0$, so, by (4.3) again, $-z \in C$. Let $z' = \frac{1}{2}v + \frac{1}{2}z \in (v,w]$. We have $v = 2z' - z$. Since both z' and $-z$ are in C, and C is a convex cone, it follows that $v \in C$, as desired. We are thus reduced to the case where $z \succ_V 0$ for every $z \in (v,w]$.

Now we claim that there exists $\lambda_0 > 0$ and $x_0, x_1, x_2 \in M$ such that $v = \lambda_0(x_1 - x_0)$ and $w = \lambda_0(x_2 - x_0)$. Since $M \subset V$ is an efficient embedding, using (4.1) we can write $v = \lambda(x-y)$ and $w = \mu(s-t)$ for some $\lambda, \mu > 0$ and $x, y, s, t \in M$. Set $\beta = \lambda/(\lambda+\mu)$, so $1 - \beta = \mu/(\lambda+\mu)$. The claim is easily verified with

$$\lambda_0 = \lambda + \mu, \quad x_0 = \beta y + (1-\beta)t, \quad x_1 = \beta x + (1-\beta)t,$$
$$x_2 = \beta y + (1-\beta)s.$$

Any $z \in (v,w]$ can be written as $z = (1-\alpha)v + \alpha w$, with $\alpha \in (0,1]$. It follows that

$$z = (1-\alpha)\lambda_0(x_1 - x_0) + \alpha\lambda_0(x_2 - x_0) = \lambda_0((1-\alpha)x_1 + \alpha x_2 - x_0).$$

Since, as in the first step, $z \succ_V 0$, it follows that $(1-\alpha)x_1 + \alpha x_2 \succ_V x_0$. Then, by (4.2), $(1-\alpha)x_1 + \alpha x_2 \succ x_0$. This holds for all $\alpha \in (0,1]$, so MC gives us $x_1 \succsim x_0$. Therefore, by (4.3), $v = \lambda_0(x_1 - x_0) \in C$.

Conversely, suppose that C is algebraically closed. To show that $\succsim$ satisfies MC, suppose that $\alpha x + (1-\alpha)y \succ z$ for all $\alpha \in (0,1]$. Then by (4.2), $\alpha(x-z) + (1-\alpha)(y-z) \in C$ for all such α. Since C is algebraically closed, it follows that $y - z \in C$. By (4.2), $y \succsim z$, validating MC.

For (iii), let V have the weak topology. Suppose first that $\succsim$ has a mixture-preserving multi-representation $\mathcal{U}$. Then (4.4) presents C as the intersection of closed sets, so it is closed.

Conversely, suppose that C is closed. If $C = V$, then by (4.3) $\succsim$ is the indifference relation, which has a mixture-preserving multi-representation consisting of a single constant function. Assume then $C \neq V$. The weak topology on V is locally convex, so by the strong separating hyperplane theorem (Aliprantis and Border, 2006, Cor. 5.84), for any $v \notin C$, there exists a linear functional $L_v \colon V \to \mathbb{R}$ such that $L_v(C) \subset [0,\infty)$ and $L_v(v) < 0$. Let $\mathcal{L} = \{L_v : v \notin C\}$. Then by (4.2),

$$x \succsim y \iff x - y \in C \iff L(x) \geq L(y) \text{ for all } L \in \mathcal{L}.$$

It follows that the restriction of $\mathcal{L}$ to M is a mixture-preserving multi-representation of $\succsim$. □

Proof of Proposition 4.2. We show that the cone K defined in Example 4.3 is algebraically closed but not closed (recall that V has the weak topology).

As a first step, we show that, for any finite, non-empty $A \subset B_1$, the subcone $K \cap \operatorname{span}(A \cup \{b_0\})$ of K is algebraically closed. Any convex cone generated by finitely many elements is algebraically closed (see e.g. Ok, 2007, G.1.6, Thm. 1), so it suffices to prove

$$K \cap \operatorname{span}(A \cup \{b_0\}) = \operatorname{cone}\{y_{A'} + b_0 : A' \neq \varnothing, A' \subset A\}. \tag{C.1}$$

The inclusion of the right-hand side in the left is obvious. Conversely, suppose v is a member of the left-hand side. We may assume $v \neq 0$. Since $v \in K$, it may be written

$$v = \sum_{k=1}^{n} \lambda_k (y_{A_k} + b_0) \tag{C.2}$$

where n is a positive integer, each coefficient λ_k is strictly positive, and each A_k is a finite, nonempty subset of B_1. It follows that v is a linear combination, with all coefficients strictly positive, of every member of $\bigcup_{k=1}^{n} A_k \cup \{b_0\}$. Since $v \in \operatorname{span}(A \cup \{b_0\})$, this is only possible if $A_k \subset A$ for each k. Therefore (C.2) presents v as a member of the right-hand side of (C.1).

We can now show that K itself is algebraically closed. Suppose given a half-open line segment $(v_0, v_1] \subset K$; we have to show $v_0 \in K$. We can find a finite set of basis elements $A \subset B_1$ such that $v_0, v_1 \in \operatorname{span}(A \cup \{b_0\})$, and therefore such that $(v_0, v_1] \subset \operatorname{span}(A \cup \{b_0\})$. Since $K \cap \operatorname{span}(A \cup \{b_0\})$ is algebraically closed, it contains v_0; therefore $v_0 \in K$, as desired.

Finally, we show that K is not closed. In this proof, let $\overline{K}$ denote the closure of K. Note that $b_0 \notin K$; we show that b_0 is nonetheless in $\overline{K}$. Suppose for a contradiction $b_0 \notin \overline{K}$. By the strong separating hyperplane theorem there exists a linear functional $f: V \to \mathbb{R}$ such that $f(b_0) < 0$ but $f(K) \subset [0, \infty)$. Now, since B_1 is uncountable, there exists some $n \in \mathbb{N}$ for which there are infinitely many $b \in B_1$ with $f(b) < n$. Let A be a nonempty, finite set of such b. Then $f(y_A) < |A|^{-2} \sum_{b \in A} n = n/|A|$. Therefore $f(y_A + b_0) < f(b_0) + n/|A|$. Since $|A|$ may be chosen to be arbitrarily large, and $f(b_0) < 0$, we can find some y_A such that $f(y_A + b_0) < 0$, contrary to $f(K) \subset [0, \infty)$. We conclude that $b_0 \in \overline{K}$. □

Proof of Proposition 4.4. Let C be an algebraically closed convex subset of a vector space V. We may assume C is nonempty; we want to show it is closed when V is endowed with the weak topology.

First consider the case when $\dim V$ is finite. The weak topology on V is then the same as the Euclidean topology. The following argument is based on Holmes (1975, §11A(c)). We use the fact that C, like any convex subset in a finite-dimensional vector space, has a non-empty relative interior $\operatorname{ri} C$ (Aliprantis and Border, 2006, Lemma 7.33). This is an open subset of $\operatorname{aff} C$. Translating C, we can assume that $0 \in \operatorname{ri} C$, in which case $\operatorname{aff} C = \operatorname{span} C$. Let x be in the closure of C, which is contained in $\operatorname{aff} C$. For any $\alpha \in (0, 1)$, $X = -\frac{1-\alpha}{\alpha} \operatorname{ri} C$ is open in $\operatorname{aff} C$, so $x + X$ contains a point $x' \in C$. Then

$$\alpha x \in \alpha(x' - X) = \alpha x' + (1 - \alpha) \operatorname{ri} C \subset C.$$

Thus $(x, 0] \subset C$. Since C is algebraically closed, $x \in C$; thus C is closed.

Now suppose V has countable dimension. By definition, a subset X of V is closed in the *finite* topology on V if and only if $X \cap W$ is closed in the Euclidean topology in every finite-dimensional subspace W of V. Since, for each finite-dimensional $W \subset V$, $C \cap W$ is algebraically closed, the preceding argument shows that C is closed in the finite topology. By a result due to Klee (1953), but stated more fully in Kakutani and Klee (1963), the finite topology on a countable dimensional vector space makes it a locally convex topological vector space. By another version of the strong separating hyperplane theorem (Aliprantis and Border, 2006, Cor. 5.80), C is the intersection of half-spaces that are closed in the weak topology. C itself is therefore closed in the weak topology. □

The proof of Proposition 4.5 will use the following observation, given an efficient embedding $M \subset V$ of a preordered mixture space.

Lemma C.1. *Suppose given $(s, t) \in \Gamma_{\succsim}$, $x, y \in M$, and $\mu > 0$. The following are equivalent:*

(i) There exists $\lambda > 0$ such that $\lambda(x - y) \succsim_V \mu(s - t)$.

(ii) We have $(x, y) \in \Gamma_{\succsim}$, and (x, y) weakly dominates (s, t).

Proof. We repeatedly use facts (4.2) and (4.3) about efficient embeddings. Suppose (i) holds. We have $(s, t) \in \Gamma_{\succsim} \implies s \succsim t \implies \mu(s-t) \succsim_V 0 \implies \lambda(x-y) \succsim_V 0 \implies x \succsim y \implies (x, y) \in \Gamma_{\succsim}$. Rearranging the inequality in (i), and setting $\alpha = \lambda/(\lambda + \mu)$, we find $\alpha x + (1 - \alpha)t \succsim \alpha y + (1 - \alpha)s$. Therefore (x, y) weakly dominates (s, t). Thus (ii) holds. Conversely, given (ii), we have $\alpha x + (1 - \alpha)t \succsim \alpha y + (1 - \alpha)s$ for some $\alpha \in (0, 1)$. Rearranging, we obtain $\lambda(x - y) \succsim_V \mu(s - t)$ with $\lambda = \alpha\mu/(1 - \alpha)$. Thus (i) holds. □

Proof of Proposition 4.5. For (i), it is a standard result that the algebraic interior of a convex cone consists of its order units; see e.g. Aliprantis and Tourky (2007, Lemma 1.7). The proof of (i) essentially translates this fact into a result about M itself. We will rely on the basic facts (4.2) and (4.3) about efficient embeddings without further comment.

Suppose SD holds with respect to some $(x, y) \in \Gamma_{\succsim}$. Let $v = x - y \in C$. We note that, since C is a convex cone, $\operatorname{aff}(C) = \operatorname{span}(C) = C - C$. Thus, given $w \in \operatorname{aff}(C)$, we can write $w = w_1 - w_2$ with $w_1, w_2 \in C$. Since $w_2 \in C$, we also have $w_2 = \mu(s - t)$ for some $\mu > 0$ and $(s, t) \in \Gamma_{\succsim}$. By SD, (x, y) weakly dominates (s, t). So there exists, by Lemma C.1, some $\lambda > 0$ such that $\lambda v = \lambda(x - y) \succsim_V \mu(s - t) = w_2$. Therefore $v - \frac{1}{\lambda} w_2 \in C$. Since also $\frac{1}{\lambda} w_1 \in C$, we find that $v + \frac{1}{\lambda} w_1 - \frac{1}{\lambda} w_2 = v + \frac{1}{\lambda} w \in C$. Since C is convex, we deduce $[v, v + \frac{1}{\lambda} w] \subset C$. Since $w \in \operatorname{aff}(C)$ was arbitrary, this shows v is in the relative algebraic interior $\operatorname{rai}(C)$.

Conversely, suppose that $\operatorname{rai}(C)$ is nonempty. Fix $v \in \operatorname{rai}(C)$; then $v = \lambda(x - y)$ for some $\lambda > 0$ and $x \succsim y$. Given any $(s, t) \in \Gamma_{\succsim}$, we have $t - s \in -C \subset \operatorname{aff}(C)$. For some $\epsilon > 0$, we must have $v + \epsilon(t - s) \in C$, so $\lambda(x - y) \succsim_V \epsilon(s - t)$. By Lemma C.1, we have $(x, y) \in \Gamma_{\succsim}$ and (x, y) weakly dominates (s, t). Therefore this (x, y) weakly dominates every $(s, t) \in \Gamma_{\succsim}$, so SD holds.

For (ii), suppose CD holds, so that every $(s, t) \in \Gamma_{\succsim}$ is weakly dominated by an element of some countable set $D \subset \Gamma_{\succsim}$. Let $S = \{n(x - y) : n \in \mathbb{N}, (x, y) \in D\} \subset C$. Since D is countable, so is S. We claim S is cofinal in C. Let $w \in C$. We can write $w = \mu(s - t)$ with $\mu > 0$, $s \succsim t$. Some $(x, y) \in D$ weakly dominates (s, t). Therefore, by Lemma C.1, there exists $\lambda > 0$ with $\lambda(x - y) \succsim_V \mu(s - t) = w$. Choose an integer $n > \lambda$. Then $n(x - y) \succsim_V \lambda(x - y) \succsim_V w$. Since $n(x - y) \in S$, S is cofinal in C.

Conversely, suppose that S is a countable set, cofinal in C. For each $v \in S$, we can choose $\lambda_v > 0$ and $x_v, y_v \in M$ with $x_v \succsim y_v$ such that $v = \lambda_v(x_v - y_v)$. Let $D = \{(x_v, y_v) : v \in S\}$. Since S is countable, so is D. To prove CD, we show that every $(s, t) \in \Gamma_{\succsim}$ is weakly dominated by an element of D. Since $s \succsim t$, we have $s - t \in C$. Since S is cofinal, there exists $v \in S$ such that $v \succsim_V s - t$. It follows from Lemma C.1 that (x_v, y_v) weakly dominates (s, t). □

Proof of Corollary 4.9. By Proposition 4.5(ii), it suffices to show that there is a countable set cofinal in C if and only if there is a countable-dimensional subspace cofinal in $\operatorname{span}(C)$.

Suppose $S \subset C$ is countable and cofinal. Let $Z = \operatorname{span}(S)$. Because C is a convex cone, any $v \in \operatorname{span}(C)$ can be written in the form $v = x - y$ with $x, y \in C$. There is some $s \in S$ such that $s \succsim_V x$; but then $s \succsim_V v$. Since $s \in Z$, Z is cofinal in $\operatorname{span}(C)$. It has countable dimension since S is countable.

Conversely, suppose a countable-dimensional subspace Z is cofinal in $\operatorname{span}(C)$. Let $b_1, b_2, \ldots$ be a countable (finite or infinite) basis for Z. Since $b_i \in \operatorname{span}(C)$, it can be written as $x_i - y_i$ with $x_i, y_i \in C$. Note that $x_i \succsim_V b_i$. Let S consist of all linear combinations of the x_i with non-negative integer coefficients; it is a countable subset of C. Let $v \in C$. There exists $z \in Z$ such that $z \succsim_V v$. We can write z as a finite sum $z = \sum_i \lambda_i b_i$, for some $\lambda_i \in \mathbb{R}$. If λ is a positive integer greater than all the λ_i, then $S \ni \sum_i \lambda x_i \succsim_V z \succsim_V v$. Therefore S is cofinal in C. □

Proof of Lemma 4.10. The first claim, at least, is well-known; Bosi and Herden (2016), for example, provide two proofs of the first implication. But we give the short proofs for convenience.

To show *CMR* $\implies$ *Cl*, suppose $\mathcal{U}$ is a continuous mixture-preserving multi-representation of $\succsim$. For each $u \in \mathcal{U}$, define $\tilde{u}: M^2 \to \mathbb{R}$ by $\tilde{u}(x, y) = u(x) - u(y)$. This $\tilde{u}$ is continuous, and $\Gamma_{\succsim} = \bigcap_{u\in\mathcal{U}} \tilde{u}^{-1}([0, \infty))$. Thus $\Gamma_{\succsim}$ is the intersection of closed sets, so Cl holds.

To show *Cl* $\implies$ *Con*, assume that $\Gamma_{\succsim}$ is closed. Let $x \in M$. The map $f_x: M \to M^2$ given by $f_x(y) = (y, x)$ is continuous. Therefore, $\{y : y \succsim x\} = {f_x}^{-1}(\Gamma_{\succsim})$ is closed. A similar argument shows that $\{y : x \succsim y\}$ is closed. Hence Con holds.

For the second claim of the lemma, suppose M is a mixture space and the maps $f_{x,y}$ are continuous. To show *Con* $\implies$ *MC*, suppose that $\succsim$ is continuous. Suppose that $x\alpha y \succ z$ for all $\alpha \in (0, 1]$. Since $\{w : w \succsim z\}$ is closed, so is ${f_{x,y}}^{-1}(\{w : w \succsim z\})$. The latter contains $(0, 1]$, so it also contains 0. Thus $y \succsim z$, establishing MC. □

The next lemma records some basic facts about the weak topology that will be used in the proof of Proposition 4.11.

Lemma C.2. *Let M_1 and M_2 be mixture spaces, each with the weak topology.*

(i) Suppose $f: M_1 \to M_2$ is mixture-preserving. Then f is continuous.
(ii) The weak topology on $M_1 \times M_2$ equals the product topology.[26]
(iii) If M_1 is a mixture subspace of M_2, then it is a topological subspace.
(iv) If M_2 is a vector space and $M_1 \subset M_2$ is a linear subspace, then M_1 is closed in M_2.

Proof. (i) By definition of the weak topology on M_2, a function $f: X \to M_2$ from an arbitrary topological space X is continuous if and only if $g \circ f$ is continuous for every mixture-preserving $g: M_2 \to \mathbb{R}$. Our $f: M_1 \to M_2$ is mixture preserving, so $g \circ f$ is mixture-preserving, and therefore continuous on M_1.

(ii) The weak topology on $M_1 \times M_2$ is the coarsest one such that every mixture-preserving $f: M_1 \times M_2 \to \mathbb{R}$ is continuous. The product topology is the coarsest one such that the projections π_i of $M_1 \times M_2$ onto M_i are continuous. Equivalently, it is the coarsest one such that for all mixture-preserving $f_1: M_1 \to \mathbb{R}$ and $f_2: M_2 \to \mathbb{R}$, the function $f_1 \circ \pi_1 + f_2 \circ \pi_2: M_1 \times M_2 \to \mathbb{R}$ is continuous. Since the latter function is clearly mixture-preserving, it suffices to show that (conversely) every mixture-preserving f is of this form.

Fix $z_1 \in M_1$ and $z_2 \in M_2$. For $x_i \in M_i$ define $f_1(x_1) = f(x_1, z_2)$ and $f_2(x_2) = f(z_1, x_2) - f(z_1, z_2)$. It is easy to check that f_1, f_2 so defined are mixture preserving. Moreover, using the mixture-preservation property of f,

$$\begin{aligned} f_1(x_1) + f_2(x_2) - f(x_1, x_2) &= f(x_1, z_2) + f(z_1, x_2) \\ &\quad - (f(z_1, z_2) + f(x_1, x_2)) \\ &= 2f(x_1 \tfrac{1}{2} z_1, z_2 \tfrac{1}{2} x_2) - 2f(z_1 \tfrac{1}{2} x_1, z_2 \tfrac{1}{2} x_2) \\ &= 0. \end{aligned}$$

Therefore $f_1 \circ \pi_1 + f_2 \circ \pi_2 = f$, as desired.

(iii) The claim is that the weak topology on M_1 coincides with the subspace topology inherited from M_2. The restriction to M_1 of a mixture-preserving function on M_2 is mixture preserving; it follows that the subspace topology on M_1 is contained in its weak topology. To show the converse, it suffices to show that any mixture-preserving $M_1 \to \mathbb{R}$ extends to a mixture-preserving function $M_2 \to \mathbb{R}$. To prove this using standard facts from linear algebra, we can first embed M_2 as a convex set in a vector space V (see Section 4.1); thus M_1 is also a convex subset of V. Any mixture-preserving function $f: M_1 \to \mathbb{R}$ extends to an affine (i.e. linear plus constant) function on V; the restriction of this affine function to M_2 is a mixture-preserving extension of f.

(iv) For any $x \in M_2 \setminus M_1$, there is a linear (hence mixture-preserving) function $g: M_2 \to \mathbb{R}$ such that $g(M_1) = \{0\}$ and $g(x) = 1$. Then $g^{-1}((0, \infty))$ is an open neighbourhood of x disjoint from M_1. Thus $M_2 \setminus M_1$ is open and M_1 is closed in M_2. □

Proof of Proposition 4.11. We first show that the mixture preorder $\succsim$ defined in Example 4.12 is continuous. Fix $(v, w) \in M$. Let $U = \{(x, y) : (x, y) \succsim (v, w)\}$ and $L = \{(x, y) : (v, w) \succsim (x, y)\}$. We need to show that U and L are closed in M, which has the weak topology. The two cases are similar, so we consider the former.

Let $K_v = K \cap V_v$. Define a function $f: M \to V \times V$ by $(x, y) \mapsto (x, y) - (v, w)$. It follows from (4.5) that $U = f^{-1}(\{0\} \times K_v)$. Give $V \times V$ the weak topology. Since f is mixture-preserving, Lemma C.2(i) tells us that f is continuous. So, to show that U is closed, it suffices to show that $\{0\} \times K_v$ is closed in $V \times V$.

In the first step of proving Proposition 4.2 we showed that K_v, that is, $K \cap \mathrm{span}(A_v \cup \{b_0\})$, is an algebraically closed convex cone. Thus $\{0\} \times K_v$ is an algebraically closed convex subset of $\{0\} \times V_v$. Since V_v, and hence $\{0\} \times V_v$, is a finite-dimensional vector space, Proposition 4.4 implies that $\{0\} \times K_v$ is closed in the weak topology on $\{0\} \times V_v$.

By Lemma C.2(iii), $\{0\} \times V_v$, with the weak topology, is a topological subspace of $V \times V$. Moreover, it is a closed subspace, by Lemma C.2(iv). In summary, $\{0\} \times K_v$ is closed in a closed subspace of $V \times V$; therefore it is closed in $V \times V$.

We now show that $\Gamma_{\succsim}$ is not closed in $M \times M$. Note that $z = (0, b_0; 0, 0)$ is an element of $M \times M$, but not of $\Gamma_{\succsim}$. It suffices to show that z is in the closure of $\Gamma_{\succsim}$ in $M \times M$. Therefore, it suffices to find a net (z_α) in $\Gamma_{\succsim}$ converging to z in $M \times M$. Here M has the weak topology and $M \times M$ has the resulting product topology. Similarly, give V the weak topology, and $V^2 \times V^2$ the product topology. By Lemma C.2(ii), both these product topologies are again the weak topologies; Lemma C.2(iii) then implies that $M \times M$ is a topological subspace of $V^2 \times V^2$. So it will suffice that (z_α) converges to z in $V^2 \times V^2$.

Recall that b_0 is in the closure $\overline{K}$ of K in V, as proved as the last step in the proof of Proposition 4.2. Let (y_α) be a net in K converging to b_0. Note that, by definition, $K \subset \mathrm{cone}(B) = V^+$. Therefore each y_α can be written as $y_\alpha = x_\alpha + \lambda_\alpha b_0$, with $x_\alpha \in \mathrm{cone}(B_1)$ and $\lambda_\alpha \geq 0$. Note $x_\alpha \in V^+$ and $y_\alpha \in V_{x_\alpha}$, so (x_α, y_α) is in M. Moreover, by (4.5), $(x_\alpha, y_\alpha) \succsim (x_\alpha, 0)$. Therefore $z_\alpha := (x_\alpha, y_\alpha; x_\alpha, 0)$ is in $\Gamma_{\succsim}$.

Now, *any* element of V can be written uniquely in the form $y = x + \lambda b_0$ with $x \in \mathrm{span}(B_1)$ and $\lambda \in \mathbb{R}$. Define a linear map $f: V \to V^2 \times V^2$ by $f(y) = (x, y; x, 0)$. Note $z_\alpha = f(y_\alpha)$. Since, by Lemma C.2(i), f is continuous, we have $\lim_\alpha z_\alpha = f(b_0) = z$. □

Proof of Proposition 5.1. It is obvious that a preorder satisfying SMR satisfies MR and admits a strictly increasing mixture-preserving function. (Note that we require multi-representations to be nonempty.) Conversely, let $u': M \to \mathbb{R}$ be mixture-preserving and strictly increasing, and $\mathcal{U}$ be a mixture-preserving multi-representation. Let $\mathcal{U}' = \{u' + nu : n \in \mathbb{N}, u \in \mathcal{U}\}$. First, note that for any $n \in \mathbb{N}$ and $u \in \mathcal{U}$, $u' + nu$ is strictly increasing. Now suppose that $u'(x) + nu(x) \geq u'(y) + nu(y)$ for all $n \in \mathbb{N}$, $u \in \mathcal{U}$. Since, for each u, n can be arbitrarily large, we must have $u(x) \geq u(y)$. Since $\mathcal{U}$ is a multi-representation, we find $x \succsim y$, so $\mathcal{U}'$ is a mixture-preserving multi-representation containing only strictly increasing functions. □

Proof of Lemma 5.3. Let $M \subset V$ be an efficient embedding, with positive cone $C \subset V$. Suppose given a mixture-preserving multi-representation $\mathcal{U}$. For each $u \in \mathcal{U}$, let $\tilde{u}$ be its extension to an

[26] Here $M_1 \times M_2$ is a mixture space with respect to the component-wise mixing operation: $(x_1, x_2)\alpha(y_1, y_2) = (x_1\alpha y_1, x_2\alpha y_2)$.

affine function $V \to \mathbb{R}$, and let A_u be the open half-space $A_u = \{v \in V : \tilde{u}(v) < \tilde{u}(0)\}$. It follows from (4.4) that $\mathcal{A} = \{A_u : u \in \mathcal{U}\}$ is an open cover of $V \setminus C$, in the weak topology on V.

Consider first the case where $\dim M$ is finite, and hence, by Proposition 4.1(i), $\dim V$ is finite. Then the weak topology on V coincides with the Euclidean topology, and V is a second-countable topological space, as is its topological subspace $V \setminus C$. By Lindelöf's lemma, $\mathcal{A}$ contains a countable subcover $\mathcal{A}'$. We can write $\mathcal{A}' = \{A_u : u \in \mathcal{U}'\}$ for some countable subset $\mathcal{U}' \subset \mathcal{U}$. Then

$$C = \bigcap_{u \in \mathcal{U}'} \{v \in V : \tilde{u}(v) \geq \tilde{u}(0)\}. \tag{C.3}$$

It follows from (4.4) that $\mathcal{U}'$ is a mixture-preserving multi-representation of $\succsim$. Finally we note that $|\mathcal{U}'| = \aleph_0 \leq \max(\aleph_0, \dim M)$.

Now suppose $\dim M = \dim V = \kappa$ for some infinite cardinal κ. Let B be a basis of V, and let $\mathcal{P}$ be the set of finite subsets of B; note that $|\mathcal{P}| = \kappa$. For each $P \in \mathcal{P}$, $\mathcal{A}_P := \{A_u \cap \operatorname{span} P : u \in \mathcal{U}\}$ is an open cover of $\operatorname{span} P \setminus C$ in the weak topology on $\operatorname{span} P$. As in the previous paragraph, it contains a countable subcover $\mathcal{A}'_P$, which we can write in the form $\mathcal{A}'_P = \{A_u \cap \operatorname{span} P : u \in \mathcal{U}'_P\}$, with $\mathcal{U}'_P \subset \mathcal{U}$ countable. Let $\mathcal{U}' = \bigcup_{P \in \mathcal{P}} \mathcal{U}'_P$. Choose any $v \in V \setminus C$. It is in $\operatorname{span} P$ for some P, and therefore it is in A_u for some $u \in \mathcal{U}'$. So $\mathcal{U}' = \{A_u : u \in \mathcal{U}'\}$ is an open cover of $V \setminus C$. For the same reason as before, $\mathcal{U}'$ is a mixture-preserving multi-representation of $\succsim$. Finally, since $|\mathcal{P}| = \kappa$ and each $\mathcal{U}'_P$ is countable, $|\mathcal{U}'| = \kappa \leq \max(\aleph_0, \dim M)$. □

Proof of Proposition 5.2. For (i), assume that $\dim M$ is countable and let $\succsim$ be a mixture preorder on M that has a mixture-preserving multi-representation; we have to show that it has one using only strictly increasing functions. Let $M \subset V$ be an efficient embedding, so, by Proposition 4.1(i), $\dim V$ is countable. Since $V = \operatorname{span} M$, we can pick a (finite or countably infinite) basis $B = \{v_1, v_2, \ldots\} \subset M$ of V. By Lemma 5.3, $\succsim$ has a finite or countably infinite mixture-preserving multi-representation $\mathcal{U} = \{u_1, u_2, \ldots\}$. Let $\tilde{u}_i : V \to \mathbb{R}$ be the unique extension of u_i to an affine function; thus $L_i := \tilde{u}_i - \tilde{u}_i(0)$ is a linear functional on V. Rescaling the u_i as necessary, we can assume $|L_i(v_j)| \leq 1$ whenever $j \leq i$. We define a mixture-preserving function u on M by

$$u(x) = \sum_{i=1}^{|\mathcal{U}|} 2^{-i} L_i(x).$$

This is clearly well-defined when $|\mathcal{U}|$ is finite. If $|\mathcal{U}|$ is infinite, note that every $x \in M$ can be written in the form $x = \sum_{j=1}^{|B|} c_j v_j$, with finitely many nonzero $c_j \in \mathbb{R}$. It follows that $|L_i(x)| \leq \sum_{j=1}^{|B|} |c_j||L_i(v_j)| \leq \sum_{j=1}^{|B|} |c_j|$, for all sufficiently large i. Therefore the sum defining $u(x)$ is absolutely convergent, making u a well-defined mixture-preserving function. It is also strictly increasing. By Proposition 5.1, $\succsim$ has a mixture-preserving multi-representation using only strictly increasing functions.

For part (ii), we show that the mixture preorder defined in Example 5.4 satisfies MR but not SMR.

That preorder was defined by a mixture-preserving multi-representation, so it satisfies MR. We show that it does not admit any strictly-increasing function $M \to \mathbb{R}$. Suppose for contradiction that u is such a function. In the notation of the example, for each $\alpha < \kappa$, define $f(\alpha) = -u(v_\alpha)$. Given $\alpha < \beta < \kappa$, we have $v_\alpha \succ v_\beta$, and hence $u(v_\alpha) > u(v_\beta)$. This shows that f is a strictly increasing function of α, and hence there are uncountably many intervals $(f(\alpha), f(\alpha+1)) \subset \mathbb{R}$ that are nonempty, pairwise disjoint, and open. But that is impossible: each open interval must contain a rational number, of which there are countably many. □

Proof of Proposition 5.5. Suppose a preorder $\succsim$ on M is represented by $\mathcal{U} \subset M^*$. Let $(M^*)^+ \subset M^*$ consist of the functions in M^* that are increasing with respect to $\succsim$. Write $\mathcal{K} = \operatorname{cone}(\mathcal{U} \cup \mathcal{C})$. To prove the Proposition, it is sufficient to show that $\overline{\mathcal{K}} = (M^*)^+$.

We first verify $\overline{\mathcal{K}} \subset (M^*)^+$. It is obvious that $\mathcal{K} \subset (M^*)^+$. Suppose (f_α) is a net in $\mathcal{K}$ converging to f, and suppose $x \succsim y$. Then $f_\alpha(x) \geq f_\alpha(y)$ for all α. Since M^* has the topology of pointwise convergence, $\lim_\alpha f_\alpha(x) = f(x)$ and $\lim_\alpha f_\alpha(y) = f(y)$; therefore $f(x) \geq f(y)$. Thus f is increasing, i.e. $f \in (M^*)^+$.

Conversely, to show $(M^*)^+ \subset \overline{\mathcal{K}}$, we first embed M in M^{**}, the algebraic dual of M^*, via the mapping $\phi: M \to M^{**}$ given by $\phi(x)(f) = f(x)$. It is easy to check that ϕ is mixture-preserving (it is also injective, as shown in Mongin (2001), but we do not use this). The subspace $\operatorname{span}(\phi(M)) \subset M^{**}$ separates the points of M^*, so $(M^*, \operatorname{span}(\phi(M)))$ is a dual pair of vector spaces. Moreover, the topology on M^* is the weak topology with respect to this pairing, so it follows from the fundamental theorem of duality (Aliprantis and Border, 2006, Thm. 5.93) that $\operatorname{span}(\phi(M))$ is the continuous dual of M^*.

Suppose for a contradiction that $f \in (M^*)^+$ but $f \notin \overline{\mathcal{K}}$. The vector space M^* is locally convex, and since $\mathcal{K}$ is a convex cone, we may use the strong separating hyperplane theorem (Aliprantis and Border, 2006, Cor. 5.80) to obtain $F \in \operatorname{span}(\phi(M))$ such that $F(\overline{\mathcal{K}}) \subset [0, \infty)$ and $F(f) < 0$. Write $F = \sum_{x \in M} \lambda_x \phi(x) - \sum_{x \in M} \mu_x \phi(x)$ for nonnegative $\lambda_x, \mu_x \in \mathbb{R}$, only finitely many nonzero. Since ϕ is mixture preserving, we can combine terms to obtain $F = \lambda\phi(x) - \mu\phi(y)$ for some nonnegative $\lambda, \mu \in \mathbb{R}$, and $x, y \in M$. Since F is nonnegative on $\overline{\mathcal{K}}$, and hence on the constant functions, we must have $\lambda = \mu$. Thus $F(f) = \lambda(f(x) - f(y)) < 0$. Since f is increasing, it follows that $x \not\succsim y$. Thus for some $g \in \mathcal{U}$, $g(x) < g(y)$, implying that $F(g) < 0$. This is impossible since $g \in \overline{\mathcal{K}}$. □

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